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A MISUNDERSTANDING IN INDEX-NUMBER THEORY: THE TRUE KONÜS CONDITION ON COST-OF-LIVING INDEX NUMBERS AND ITS LIMITATIONS

By HENRY SCHULTZ*

IN THE RECENT literature on index numbers the so-called "Konüs inequalities," within which the "true" index of the cost of living is supposed to fall, have played an important role. Konüs' paper was published in Russian in 1924,[1] and thus far our only knowledge of it has been the incidental, though appreciative, observations regarding it in Bortkiewicz's review of Haberler's book on index numbers published in 1928.[2] It is this inadequate summary of Bortkiewicz which Staehle used in 1934 in his important work on international comparisons of cost of living[3] and which constituted a point of departure for his own researches.

From my first reading of Dr. Staehle's manuscript I got the feeling that there was more to the Konüs condition than was evident from the Staehle-Bortkiewicz statement of it, but could not afford the time to look into the matter. In 1934–35, however, I was called upon to prepare a few lectures on the bearing of the modern theory of utility and exchange on the problem of index numbers, and I decided to look into the original paper by Konüs. Not being able to read Russian, I had a

* We are very sorry to report that Professor Schultz and his family were killed in an automobile accident near San Diego, California, on November 26, 1938.

The proof of this paper and of the following one by Professor A. A. Konüs did not reach Professor Schultz before his death. We are greatly indebted to Dr. Abraham Wald, Cowles Commission for Research in Economics, with whom Professor Schultz had discussed the subject, for reading proof of both papers; and to Mr. Jacob L. Mosak, assistant to Professor Schultz, for supplying certain modifications which the latter had intended to make in the proof.—MANAGING EDITOR.

[1] "The Problem of the True Index of the Cost of Living," *The Economic Bulletin of the Institute of Economic Conjuncture*, Moscow, No. 9–10 (36–37), September–October, 1924, pp. 64–71.

[2] L. von Bortkiewicz, review of Gottfried Haberler's, *Der Sinn der Indexzahlen*, *Magazin der Wirtschaft*, Berlin, Vol. 4, 1928, No. 11, pp. 427–29.

[3] Hans Staehle, "International Comparison of Food Costs," *International Comparison of Cost of Living*, Studies and Reports of the International Labour Office, Series N (Statistics), No. 20, Geneva, 1934.

translation prepared of it which has been used in my classes since then.[4] This paper brought to light the following important facts:

(1) *The condition which is attributed to Konüs in the literature is but one step in his argument and it is not even stated explicitly by him. The fundamental condition to which Konüs attaches most importance was not mentioned by Bortkiewicz.*

(2) Konüs' paper contains a clear statement of the problem of index numbers of the cost of living from the point of view of the modern theory of consumers' choice, and anticipates some of the results of A. L. Bowley, R. G. D. Allen, Hans Staehle, and other writers on the subject.

(3) Konüs' contribution is subject to a most serious limitation which may be overlooked by some readers.

It is for these reasons that I have submitted the translation of Konüs' paper for publication. I feel certain that it will be welcomed by all serious students of index numbers.

Konüs starts with a clear definition of the problem: The true index of the cost of living is the ratio of the monetary expenditures of an individual which secure for him the *same standard of living* or total utility in two situations which differ only in respect to prices. He then demonstrates that Laspeyres' formula provides an upper limit for the cost-of-living index based on the standard of living in the base situation, and that Paasche's formula provides a lower limit for the cost of living based on the standard of living in the "given" situation. But since the standards of living in the two situations are not necessarily equal to each other, the indices of the cost of living may be unequal and either index may lie outside the limits provided by Laspeyres' and Paasche's formulas.[5]

Konüs then sets himself the problem of finding a condition which will insure the approximate equality of two standards of living. If such a condition can be found, then the cost-of-living index is simply the ratio of the expenditures in the two situations. *This problem is quite different from that ascribed to him in the literature, namely, the problem of finding a condition which will provide the second limit to a cost-of-living index.*

Before proceeding to summarize Konüs' conclusions and to point out their limitations, it is desirable to adopt a more convenient notation

[4] I am grateful to the following graduate students for their reports on various aspects of index-number theory: Miss Fredlyn Ramsey, Mr. Orvis Schmidt, Mr. Martin Bronfenbrenner, Mr. Jacob L. Mosak, and Mr. H. Gregg Lewis.

[5] In 1933 these findings were presented to Western readers in a very clear and elegant form by Mr. R. G. D. Allen who developed his theory quite independently of Konüs. See his, "On the Marginal Utility of Money and Its Application," *Economica*, Vol. 13, May, 1933, pp. 200–209.

than that used by him. The following notation,† which has been adopted after some experimentation, has several advantages for our purposes. (In the translation Konüs' formulas are given in this, as well as in his own notation.) Let:

Q_t^α = the bundle of goods on the expenditure curve E_t and the indifference curve ϕ^α;

q_{ti}^α = the individual components of the bundle Q_t^α;

p_{ti} = the prices of q_{ti}^α on the expenditure curve E_t;

${}_\tau R_t^\alpha = \Sigma p_{\tau i} q_{ti}^\alpha$ = the cost of the bundle Q_t^α at τ prices.

If $\tau = t$, then we have:

${}_t R_t^\alpha \equiv \Sigma p_{ti} q_{ti}^\alpha$ = the actual total expenditure on the bundle Q_t^α.

In this case we may drop the left subscript and write

$R_t^\alpha \equiv {}_t R_t^\alpha$,

$I_{t\tau}^\alpha = \dfrac{{}_\tau R_\tau^\alpha}{{}_t R_t^\alpha}$ = the index of the cost of living relating to the standard of living ϕ^α, and the price situations t and τ.

In particular let the two price situations be the "base" situation, $t=0$, and the "given" situation, $\tau=1$. Let the standards of living for the individual in the two situations be $\phi^\alpha = \phi^0$, and $\phi^\alpha = \phi^1$. Then we have:

$R_0^0 \equiv {}_0R_0^0 = \Sigma p_{0i} q_{0i}^0$ = the total expenditure in the price situation 0 on the equilibrium bundle Q_0^0;

$R_1^1 \equiv {}_1R_1^1 = \Sigma p_{1i} q_{1i}^1$ = the total expenditure in the price situation 1 on the new equilibrium bundle Q_1^1;

${}_0R_1^1 = \Sigma p_{0i} q_{1i}^1$ = the cost of the bundle Q_1^1 at p_0 prices;

${}_1R_0^0 = \Sigma p_{1i} q_{0i}^0$ = the cost of the bundle Q_0^0 at p_1 prices;

${}_1R_1^0 = \Sigma p_{1i} q_{1i}^0$ = the cost of a bundle Q_1^0 (yielding the same standard of living as Q_0^0) at p_1 prices;

${}_0R_0^1 = \Sigma p_{0i} q_{0i}^1$ = the cost of a bundle Q_0^1 (yielding the same standard of living as Q_1^1) at p_0 prices.

The relation between these six quantities may be best obtained from a diagram such as Figure 1 which relates to only two commodities (X_1) and (X_2), of which the second is taken as the *numeraire*, so that $p_{t2}=1$ in all prices situations, t. In this diagram ϕ^0 and ϕ^1 represent two indifference curves. By definition, all "bundles" (points) lying on the same indifference curve yield the same standard of living. The straight lines give by their slopes the ratio $p_{t1}/p_{t2} = p_{t1}$ and by their

† Because of mechanical limitations superscripts have been printed after subscripts instead of directly over them.—MANAGING EDITOR.

intercepts on the Oq_2-axis the total expenditures $R=\Sigma pq$, the "situation" or "year" and standard of living to which the prices and the quantities relate being indicated by the appropriate subscript and superscript.

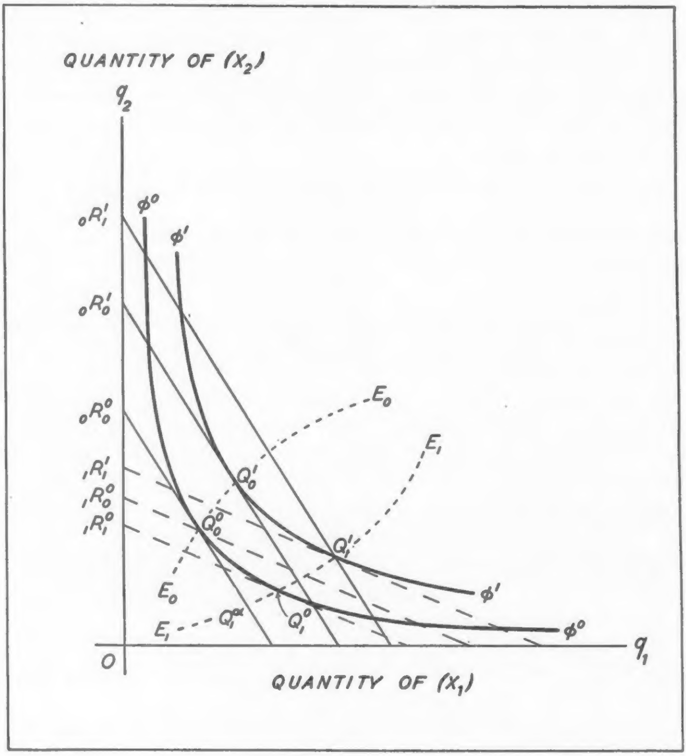

FIGURE 1.—Illustration of the definitions and relations which are useful in index-number theory.

By definition,

$I_{01}{}^0 = \dfrac{{}_1R_1{}^0}{{}_0R_0{}^0} = \dfrac{1}{I_{10}{}^0} =$ cost-of-living index relating to the standard of living ϕ^0;

$I_{01}{}^1 = \dfrac{{}_1R_1{}^1}{{}_0R_0{}^1} = \dfrac{1}{I_{10}{}^1} =$ cost-of-living index relating to the standard of living, ϕ^1;

$$P_0 = \frac{{}_1R_0{}^0}{{}_0R_0{}^0} = \text{Laspeyres' formula;}$$

$$P_1 = \frac{{}_1R_1{}^1}{{}_0R_1{}^1} = \text{Paasche's formula.}$$

With these definitions in mind, we may return to Konüs' argument. Konüs first proves that

$$\text{(1)} \qquad \begin{cases} P_0 > I_{01}{}^0 \\ P_1 < I_{01}{}^1 \end{cases}$$

(which is evident from Figure 1), and calls attention to the fact that $I_{01}{}^0$ may be different from $I_{01}{}^1$:

$$\text{(2)} \qquad I_{01}{}^0 \gtrless I_{01}{}^1,$$

and that P_0 may be different from P_1:

$$\text{(3)} \qquad P_0 \gtrless P_1,$$

(4) and neither $I_{01}{}^0$ nor $I_{01}{}^1$ need lie between P_0 and P_1.

Then he attempts to determine conditions which will insure the approximate equality of the standards of living in the two situations. Since $I_{01}{}^\alpha$ is a continuous, single-valued function of the standard of living ϕ^α, it follows that there exists a standard of living ϕ^ϵ such that its cost-of-living index $I_{01}{}^\epsilon$ lies between $I_{01}{}^0$ and $I_{01}{}^1$, and at the same time lies also between P_0 and P_1. This is possible in all of the six cases of ordering P_0, P_1, $I_{01}{}^0$, and $I_{01}{}^1$ (see p. 20), subject to the inequalities (1), (2), (3). Thus

$$\text{(5)} \qquad I_{01}{}^0 \gtrless I_{01}{}^\epsilon \gtrless I_{01}{}^1,$$

and

$$\text{(6)} \qquad P_0 \gtrless I_{01}{}^\epsilon \gtrless P_1.$$

Konüs next examines the conditions which must be satisfied if ${}_1R_1{}^1/{}_0R_0{}^0$—the ratio of the expenditures in the two periods—is to be within the same limits as the index $I_{01}{}^\epsilon$. The condition which he lays down is that

$$\text{(7)} \qquad \frac{R_1{}^1}{R_0{}^0} = \frac{{}_1R_0{}^0}{{}_0R_1{}^1},$$

or that the ratio of the expenditure in the price situation 0 to the expenditure in the price situation 1 shall be equal to the ratio of the cost of the bundle $Q_0{}^0$ priced at p_1 prices to the cost of the bundle $Q_1{}^1$ priced at p_0

prices. When this condition is satisfied, Konüs argues, the ratio of the two expenditures will lie between the same limits as $I_{01}{}^{\epsilon}$, or

$$(8) \qquad P_0 \gtrless \frac{{}_1R_1{}^1}{{}_0R_0{}^0} \gtrless P_1.$$

And he concludes, without proof, that when this condition is satisfied the two standards of living are approximately equal, and the true index of the cost of living is given by the ratio of the two expenditures.

It is important to observe that the foregoing condition differs from that imputed to Konüs in the literature on index numbers. He sought a criterion which would indicate when the two standards of living are equal. The true index of the cost of living could then be obtained unequivocally from the expenditures in the two situations; there would then be no need for determining upper and lower limits to the index. The conditions ascribed to him and used by Bortkiewciz and Staehle is designed to provide a second limit to the cost-of-living index $I_{01}{}^0$ or $I_{01}{}^1$, the upper limit to $I_{01}{}^0$ being already given by the Laspeyres formula and the lower limit to $I_{01}{}^1$ by the Paasche formula. [See equations (1).] This condition states that if the *numerators* of (7) are equal to each other, then $Q_0{}^0$ yields a lower standard of living than $Q_1{}^1$ and therefore P_0 provides an upper limit for $I_{01}{}^1$. Likewise, if the *denominators* of (7) are equal, then $Q_1{}^1$ yields a lower standard of living than does $Q_0{}^0$ and P_1 provides a lower limit for $I_{01}{}^0$.[6] These conditions were later extended by Staehle, Allen, and Lerner[7] so as to yield upper and lower limits to the cost-of-living index corresponding to *any* standard of living ϕ^α. Thus, suppose that the denominators of (7) are not equal and we wish to find a lower limit to $I_{01}{}^0$. All we have to do is to locate a bundle $Q_1{}^\alpha$ on the expenditure curve E_1E_1 (Fig. 1) and on an indifference curve ϕ^α such that $R_0{}^0 = {}_0R_1{}^\alpha$. We know then that the standard of living ϕ^α yielded by the bundle $Q_1{}^\alpha$ is lower than the standard of living ϕ^0 yielded by the bundle $Q_0{}^0$, so that ${}_1R_1{}^\alpha/{}_0R_1{}^\alpha$ is a lower limit for $I_{01}{}^0$.

I have said that Konüs states without proof that when condition (7) is satisfied, the two standards of living are approximately equal

[6] It is obvious from these considerations that the numerators of (7) cannot be equal at the same time that the denominators are equal, for this would mean that $Q_0{}^0$ yields both a lower and higher standard of living than does $Q_1{}^1$.

[7] Staehle, Hans, "A Development of the Economic Theory of Price Index Numbers," *Review of Economic Studies*, Vol. 2, June, 1935, pp. 163–188.

Allen, R. G. D., "Some Observations on the Theory and Practice of Price Index Numbers," *ibid.*, Vol. 3, Oct., 1935, pp. 57–66.

Lerner, A. P., "A Note on the Theory of Price Index Numbers," *ibid.*, pp. 50–56.

Staehle, H., Joseph, M. F. W., and Lerner, A. P., "Further Notes on Index Numbers," *ibid.*, Vol. 3, Feb., 1936, pp. 153–158.

and consequently the true index of the cost of living is given by the ratio of the two expenditures. As a matter of fact, this argument requires both a minor and a major qualification.

1. Turning first to the less important point, it should be observed that, if condition (7) is satisfied, the relation (8) should be narrowed down to the upper set of inequalities only, for it is not true even then that

$$P_0 < \frac{R_1^1}{R_0^0} < P_1.$$

Konüs' proof for (8) is as follows:

(a) Suppose that

$$\frac{R_1^1}{R_0^0} < \frac{{}_1R_0^0}{R_0^0}.$$

Then

$$R_1^1 < {}_1R_0^0,$$

and

$$\frac{R_1^1}{{}_0R_1^1} < \frac{{}_1R_0^0}{{}_0R_1^1}.$$

But from (7)

$$\frac{R_1^1}{R_0^0} = \frac{{}_1R_0^0}{{}_0R_1^1}.$$

Therefore

$$\frac{R_1^1}{{}_0R_1^1} < \frac{R_1^1}{R_0^0} < \frac{{}_1R_0^0}{R_0^0}$$

or

$$P_1 < \frac{R_1^1}{R_0^0} < P_0.$$

(b) Now suppose that

$$\frac{R_1^1}{R_0^0} > \frac{{}_1R_0^0}{R_0^0}.$$

Then

$$R_1^1 > {}_1R_0^0$$

and

$$\frac{R_1^1}{{}_0R_1^1} > \frac{{}_1R_0^0}{{}_0R_1^1}.$$

Therefore

$$\frac{R_1^1}{{}_0R_1^1} > \frac{R_1^1}{R_0^0} > \frac{{}_1R_0^0}{R_0^0}$$

or

$$P_1 > \frac{R_1^1}{R_0^0} > P_0.$$

But case (b) is not permissible. For the condition $R_1^1 > {}^1R_0^0$ implies [by (7)] that $R_0^0 > {}_0R_1^1$. But these two cannot hold simultaneously for the same reason that $R_1^1 = {}_1R_0^0$ and $R_0^0 = {}_0R_1^1$ cannot hold simultaneously. The former condition means that Q_1^1 yields a higher standard of living than does Q_0^0 whereas the latter means that Q_1^1 yields a lower standard of living than Q_0^0. We have therefore only

$$P_1 < \frac{R_1^1}{R_0^0} < P_0.$$

But since, when equation (7) is satisfied, the relation (6) must for the same reason also be narrowed down to the upper set of inequalities, the limits for I_{01}^{ϵ} and R_1^1/R_0^0 are, nevertheless, identical.

2. But—and this is of major importance—although R_1^1/R_0^0 and I_{01}^{ϵ} fall within the same limits when equation (7) is satisfied, it still does not follow that the standards of living of the base period and of the given period are equal. In fact, even if R_1^1/R_0^0 were equal to I_{01}^{ϵ}, it would follow only that the ratio of the two expenditures is equal to the ratio of the expenditures necessary to maintain the standard of living $\phi^{\alpha} = \phi^{\epsilon}$, where ϵ is intermediate between the standards $\alpha = 0$ and $\alpha = 1$. From this we are still not at liberty to conclude that the standards 0 and 1 are equal or even approximately equal. Unless some additional assumptions are made as to the nature of the utility function, the two standards may differ greatly.[8]

In a letter dated May 31, 1938, Professor Konüs accepts these criticisms and calls my attention to an article in *Mathematischeskii Sbornik*[9] (Moskva, Izdatel'stvo Akademii nauk S.S.S.R.) in which he showed that the indifference curve which satisfies condition (7) is "a hyperbola, whose center is at the origin of the co-ordinates." He then adds that "when *small* changes in price relations occur" his equation secures an approximate equality of standards of living.

[8] Of course, Konüs' entire discussion is based on the explicit assumption that utility is measurable. If the assumption is made that utility is not measurable, then we cannot talk in terms of approximate equality of standards. We can then determine only whether a given standard is higher, lower, or equal to another standard, but not by how much one standard differs from another.

[9] This is the Russian title. The French title and full citation is as follows: S. S. Buscheguennce, "Sur une classe des hypersurfaces. A propos de 'l'index idéal' de M. Irv. Fischer," *Recueil Mathématique*, XXXII, 4, 1925, Moscow.

In his own words:

"You are also right in pointing out that the equation

$$(8) \qquad \frac{K_2}{K_1} = \frac{L_2}{L_1} \qquad \left[\frac{R_1{}^1}{R_0{}^0} = \frac{{}_1R_0{}^0}{{}_0R_1{}^1} \quad (7)\right]$$

imposes extreme restrictions on the function ϕ. Indeed, an article of mine written in collaboration with Professor S. S. Buscheguennce argues that the indifference curve (in the case of two variables) which satisfies the condition (8) [or (7)] (the latter being regarded as the differential equation of this curve) is a hyperbola, whose centre is at the origin of the co-ordinates, i.e., the straight lines K_1 and K_2 when given the condition (8) [(7)] appear as tangents to the same curve of the second degree (cf. S. S. Buscheguennce, 'Sur une classe des hypersurfaces. A propos de "l'index idéal" de M. Irv. Fischer,' *Recueil Mathématique*, XXXII, 4, 1925, Moscow). Since the condition (8) [(7)] may be considered as a basis for Professor A. Bowley's price index drawn up for two equal standards of living (*Economic Journal*, June, 1928; cf. Ragnar Frisch, 'The Double Expenditure Method,' ECONOMETRICA, Vol. 4, January, 1936, p. 29) it may be deduced that, when *small* changes in price relations occur, the equation (8) [(7)] like the hyperbola, as a form of the indifference curve, secures an approximate equality of standards of living. On the other hand, it appears that a more general formula can be given for the indifference curve than the equation of the second degree."

To summarize:

Konüs' paper is a suggestive treatment of the theory of index numbers from the point of view of the mathematical theory of utility and exchange. In his clear recognition of the fact that

$$I_{01}{}^0 \gtrless I_{01}{}^1$$

and that

$$P_0 \gtrless P_1,$$

he anticipated some of the conclusions reached by R. G. D. Allen, Hans Staehle, and other writers. And in setting up conditions for securing an approximate equality of standards of living when the changes in the prices are small, he also anticipated the findings of Bowley and others.

I am grateful to Professor Jacques Bronfenbrenner for his scholarly translation, and to his son, Martin, and to my assistants, Jacob Mosak and H. Gregg Lewis, for valued assistance.

University of Chicago

THE PROBLEM OF THE TRUE INDEX OF THE COST OF LIVING[1]

By A. A. Konüs*

1. STATING THE PROBLEM

By the expression "cost of living" we mean the monetary value of those consumers' goods which are *in fact* consumed in the course of a certain period of time by an average family belonging to a given stratum of a population. Consumption of given quantities of consumers' goods defines that general state of want-satisfaction we call "the standard of living" of the family in question. Any standard of living may be reached by various combinations of quantities of consumed goods (depending on relative prices and monetary expenditures by the consumer).

If in the course of two periods of time, the general status of want-satisfaction of the family—or the "standard of living" of that family—remains constant, then we obtain the "true index of the cost of living" by dividing the cost of living at one period of time by the cost of living at the other period. This index shows the relative change occurring in the monetary cost of those consumers' goods which are necessary for the maintenance of a certain standard of living. Thus in computing the true index of the cost of living, we compare the monetary cost of two different combinations of goods which are connected solely by the condition that during the consumption of these two combinations, the general status of want-satisfaction (the standard of living) is the same.

In practice, however, the usual method of computing indexes of the cost of living is the so-called method of aggregates. This method consists of calculating the cost of a given aggregate of consumers' goods taken in amounts corresponding to the average or normal consumption[a] and at the prices prevailing at a given time, and dividing it by

[1] This article is an extract from a work as yet incomplete, on the establishment of the form of the functional dependency between consumption and prices. During this investigation, considerable aid was derived by me from the advice and suggestions of Professor S. S. Byushgens. [Note by A. A. Konüs.]

* This paper was first published by A. A. Konüs in *The Economic Bulletin of the Institute of Economic Conjuncture*, Moscow, No. 9–10 (36–37), September-October, 1924, pp. 64–71. It was translated from the Russian for Professor Henry Schultz by Dr. Jacques Bronfenbrenner of Washington University, St. Louis, Mo. Professor Schultz suggested that it be published in Econometrica. We are very glad to present in full this excellent work which so far has been known to most econometricians only through quotations by Bortkiewicz. On closer examination the paper will be found to contain many things which later have been discovered independently by others. See the further comments by Professor Schultz in this issue.—Editor.

[a] Literally: "Average norms of consumption."—Translator.

the cost of the same aggregate of consumers' goods calculated on the basis of the prices of another period.[2]

If we express the quantities of goods used in computing an index (the quantities of goods making up the before-mentioned aggregates) as:[b]

$$x_1, x_2, \cdots, x_n; \quad \| \quad q_{01}{}^0, q_{02}{}^0, \cdots, q_{0n}{}^0;$$

the prices of these goods in the first period of time (let us call it the "base period") as:

$$a_1, a_2, \cdots, a_n; \quad \| \quad p_{01}, p_{02}, \cdots, p_{0n};$$

the prices of the identical goods in another period of time (let us call it the "given period"[c]) as:

$$b_1, b_2, \cdots, b_n; \quad \| \quad p_{11}, p_{12}, \cdots, p_{1n};$$

then the index obtained by the method of aggregates (the budgetary index) will be:

$$J_b = \frac{b_1x_1 + b_2x_2 + \cdots + b_nx_n}{a_1x_1 + a_2x_2 + \cdots + a_nx_n}. \quad \| \quad P_0 = \frac{\sum p_1q_0{}^0}{\sum p_0q_0{}^0}.$$

It has already been mentioned in the literature[3] that the budgetary index, no matter how carefully it is computed, does not show exactly the changes in the cost of living, as understood in the sense outlined above. Underlying the budgetary index there is the assumption that while prices change consumption does not change. But, in reality, due to rises and falls of individual prices, consumption of the corresponding goods decreases or increases, without necessarily changing the height of the consumer's standard of living. Moreover, in computing a budgetary index, the question arises immediately, which average quantities of goods consumed should be taken as weights in computing the index. Those which were determined by an investigation of consumers' budgets in the base period, or those which were determined in the given period? It is supposed that a budgetary index computed by means of the quantities consumed during the base period would

[2] Thus for instance in calculating the budgetary index of the Central Bureau of Labor and of the Gosplan, we use as a basis a collection of 24 kinds of goods which make up the normal consumption of a worker, according to the budgetary investigations of the end of 1923.

[b] Here and in all following presentations of symbols, the first group given will be Konüs'; the second group, those which we have found more convenient.—H.S.

[c] Literally: "the period of computing the index."—TRANSLATOR.

[3] Consult, for example, L. von Bortkiewicz, "Zweck und Struktur einer Preisindexzahl," *Nordisk Statistisk Tidskrift*, Band 2, 1923, Heft 3–4.

always exaggerate somewhat the changes in the cost of living. This perfectly correct supposition is based on the following considerations: a budgetary index is nothing but an index constructed by the aid of the arithmetic mean. In such an "arithmetic mean" index, the goods that have increased most in price exert a greater influence upon the index than do the goods that have increased less in price, or than those that have become cheaper. In reality, however, we note the contrary phenomenon: consumption of the goods which have become expensive decreases, while consumption of the goods which have become cheaper increases.

Despite the evident error in the determination of a true change in the cost of living, the budgetary index was computed by the method described above, because calculation of the true index of the cost of living seemed impossible. Even were we to carry out observations on consumers, not only during the base period but during the given period as well, we should not be able to compute a true index of the cost of living, since we would have no criterion by which to isolate those groups of consumers which maintain the same standard of living through different periods of time. In order to compute a true index of the cost of living it is necessary to know which consumption-combinations of goods yield a given standard of living despite price changes.[d] Therefore the problem of constructing a true index of the cost of living is inseparably bound up with the general problem of establishing a functional relation between consumption and prices.

2. THE THEOREM RELATING TO THE INEQUALITY OF THE STANDARDS OF LIVING WHICH RESULT FROM DIFFERENT MONETARY EXPENDITURES FOR A GIVEN COMBINATION OF GOODS.[e]

Let us apply the method of the economists of the so-called mathematical school, in a somewhat modified form. Let us prove a basic theorem that allows us to establish a connection between a true index of the cost of living and a budgetary index, which relation shows itself in inequalities. However, before we proceed, let us agree on some postulates, on which all the following considerations rest.

First Postulate: The general status of want-satisfaction of a consumer (ϕ) is a function of the quantities of consumers' goods $x_1, x_2, \cdots, x_n$ consumed.

[d] This sentence is obscure in the original Russian. Possibly this rendering, too free to be called a translation, gives Konüs' meaning: " . . . it is necessary to know what changes in the quantities of goods consumed are necessary to maintain a given standard of living despite price changes."—TRANSLATOR.

[e] The word "different" is not in the original Russian. A literal translation reads " . . . results from expenditures of monetary values for a given combination of goods."—TRANSLATOR.

In economic literature the term "consumption" is used in two different senses, on the one hand in the sense of quantity ("volume of consumption"), and on the other, in the sense of purposeful operations with consumers' goods (operations leading to want-satisfaction), for example, the wearing of shoes of a certain quality, the eating of a dinner, the use of an apartment, the visiting of a motion-picture theatre, etc. It is evident that "consumption" in the first sense—the sense which interests us in this work—for one consumer is a discontinuously varying quantity (if it is at all possible to speak of "a quantity of consumption" in the case of individual consumption). Since for further considerations it is necessary that ϕ be a continuous function of $x_1, x_2, \cdots, x_n$, we shall consider statistical quantities only, since consumption "on the average, for one consumer," when computed on the basis of data referring to a sufficiently homogeneous group of consumers, may permissibly be considered a continuously varying quantity.

Similarly, it is necessary to bear in mind that subsequent propositions and conclusions are valid only when the habits of consumers, their family compositions, and their environments (exclusive of relative prices and total expenditures) do not change (i.e., function ϕ does not change its form). Thus, for instance, there is no possibility of comparing standards of living in the summer and winter months of any year, since conditions of life differ as between summer and winter. It is possible to compare only the standards of living in the summer months of one year with those of the summer months of another year.

Second Postulate: If a consumer, in the course of a certain period of time, consumed the quantities $x_1, x_2, \cdots, x_n$ of different goods, each of which had its own price $a_1, a_2, \cdots, a_n$, and spent in doing so $K_1[=R_0{}^0]$ roubles, then the quantities of the goods consumed were selected by him so that his utility function[f] was maximized:

$$\phi(x_1, x_2, \cdots, x_n) = \max = \phi(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n), \quad \Big\| \quad \phi(q_1, q_2, \cdots, q_n) = \max = \phi_0{}^0 \equiv \phi(q_{01}{}^0, q_{02}{}^0, \cdots, q_{0n}{}^0),$$

at

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = K_1 \quad \Big\| \quad \sum p_0q = R_0{}^0$$

(the bar over x and, later, over ϕ, means that the $\bar{x}$'s are selected so that ϕ will take on its maximum value $\bar{\phi}$ at

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = K_1). \quad \Big\| \quad \sum p_0q = R_0{}^0).$$

Only one combination of goods consumed determines the maximum

[f] Literal translation: " . . . function of the general condition of the satisfaction of wants."—TRANSLATOR.

satisfaction of demands at given prices and total expenditure, i.e., function ϕ has only one maximum when we are given

$$a_1, a_2, \cdots, a_n, K_1. \quad \Big\| \quad p_{01}, p_{02}, \cdots, p_{0n}, R_0^0.$$

Hence we may derive two conclusions familiar to economists.[4]

First Conclusion: The general status of want-satisfaction (standard of living) is determined by relative prices and the total expenditure of the consumer.

$$\bar{\phi} = f(a_1, a_2, \cdots, a_n, K_1). \quad \Big\| \quad \phi_0^0 = f(p_{01}, p_{02}, \cdots, p_{0n}, R_0^0).$$

Second Conclusion: The quantity of each of the goods consumed is determined by the prices of all goods and by the total expenditure of the consumer.

$$\bar{x}_i = f_i(a_1, a_2, \cdots, a_i, \cdots, a_n, K_1). \quad \Big\| \quad q_{0i}^0 = f_i(p_{01}, p_{02}, \cdots, p_{0i}, \cdots, p_{0n}, R_0^0).$$

[4] In order to maximize function $\phi(x_1, x_2, \cdots, x_n)$, given the variables $x_1, x_2, \cdots, x_n$ related by the additional condition

$$(1) \qquad K_1 = a_1x_1 + a_2x_2 + \cdots + a_nx_n, \ \Big\| \ \sum p_0q = R_0^0,$$

it is necessary, as is known, to set equal to zero the first-order partial derivatives with respect to each of the variables $x_1, x_2, \cdots, x_n$, of the function:

$$U = \phi - \lambda\left(\sum a_ix_i - K_1\right); \ \Big\| \ U = \phi - \lambda\left(\sum p_0q - R_0^0\right);$$

where λ is a coefficient which will be defined below. Thus we have n equations

$$(2) \qquad \frac{\partial\phi}{\partial x_1} - \lambda a_1 = \frac{\partial\phi}{\partial x_2} - \lambda a_2 = \cdots = \frac{\partial\phi}{\partial x_n} - \lambda a_n = 0. \ \Big\| \ \phi_{q_1} - \lambda p_{01} = \phi_{q_2} - \lambda p_{02} = \cdots = \phi_{q_n} - \lambda p_{0n} = 0.$$

These equations may be written in the following manner:

$$(2') \qquad \frac{\frac{\partial\phi}{\partial x_1}}{a_1} = \frac{\frac{\partial\phi}{\partial x_2}}{a_2} = \cdots = \frac{\frac{\partial\phi}{\partial x_n}}{a_n} = \lambda. \ \Big\| \ \frac{\phi_{q_1}}{p_{01}} = \frac{\phi_{q_2}}{p_{02}} = \cdots = \frac{\phi_{q_n}}{p_{0n}} = \lambda.$$

Equations (2′) together with equation (1) will determine all values of variables $x_1, x_2, \cdots, x_n$ giving ϕ a maximum value, as well as the coefficient λ. Thus $x_1, x_2, \cdots, x_n$ are defined as functions of the variables

$$(3) \qquad a_1, a_2, \cdots, a_n, K_1, \ \Big\| \ p_{01}, p_{02}, \cdots, p_{0n}, R_0^0,$$

so that we may assume:

$$\bar{x}_i = f_i(a_1, a_2, \cdots, a_i, \cdots, a_n, K_1); \ \Big\| \ q_{0i}^0 = f_i(p_{01}, p_{02}, \cdots, p_{0i}, \cdots, p_{0n}, R_0^0);$$

and, in accordance with this, the maximum value of the function will be a function of the very same variables (3):

$$\bar{\phi} = f(a_1, a_2, \cdots, a_n, K_1). \ \Big\| \ \phi_0^0 = f(p_{01}, p_{02}, \cdots, p_{0n}, R_0^0).$$

Let us now state and prove a theorem:

Suppose that a consumer, during a certain period of time, under a given price situation, consumed a combination of goods which determined a certain standard of living. If, in another period of time, at changed prices, the consumer spends a sum of money equal to the cost of the first combination of goods evaluated at the new prices, then he consumes a different combination of goods, which determines a standard of living higher than that which he enjoyed in the first period.

Suppose that an investigation of consumers' budgets for a certain period gave, as the actual consumption of a consumer, the quantities, $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ of different goods, at the prices $a_1, a_2, \cdots, a_n$, the total expenditure for all goods being K_1 roubles.

$$K_1 = a_1\bar{x}_1 + a_2\bar{x}_2 + \cdots + a_n\bar{x}_n. \quad \Big\| \quad R_0{}^0 = \sum p_0q_0{}^0.$$

The quantities $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ were selected by the consumer so that his general status of want-satisfaction would be a maximum one. That is

$$\begin{aligned}\phi(x_1, x_2, \cdots, x_n) &= \max \\ &= \phi(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)\end{aligned} \quad \Bigg\| \quad \begin{aligned}\phi(q_1, q_2, \cdots, q_n) &= \max \\ &= \phi(q_{01}{}^0, q_{02}{}^0, \cdots, q_{0n}{}^0)\end{aligned}$$

given

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = K_1. \quad \Big\| \quad \sum p_0q = R_0{}^0.$$

Assume that the prices of the goods consumed in another period of time change, so that they are equal, respectively, to $b_1, b_2, \cdots, b_n$. The cost of the first combination of goods (of the old aggregate) at the new prices is equal to $L_2[={}_1R_0{}^0]$ roubles,[g] i.e.,

$$b_1\bar{x}_1 + b_2\bar{x}_2 + \cdots + b_n\bar{x}_n = L_2. \quad \Big\| \quad \sum p_1q_0{}^0 = {}_1R_0{}^0.$$

Consumption at the new prices $(b_1, b_2, \cdots, b_n)$ and at the total expenditure of L_2 roubles must differ from the consumption at the prices $a_1, a_2, \cdots, a_n$ and at the expenditure K_1;[5] the quantities of goods

[g] From this point on it becomes necessary to specify more clearly than does Konüs to which standard of living the bundles and costs relate. In our notation this is made by means of superscripts. Thus ${}_1R_0{}^0$ represents the cost at p_1 prices of the bundle which lies on the base-year indifference curve (superscript 0) and the base-year expenditure curve (subscript 0).—H. S.

[5] Consumption of some goods, as a rule, may not change, when we have a particular combination of price and expenditure changes. Had the consumption of *all goods* remained unchanged, then in the equations (2′) of the preceding footnote, with reference to the new prices:

actually consumed $(\bar{x}_1', \bar{x}_2', \cdots, \bar{x}_n')$ will be selected by the consumer again in such a way that his utility function will be maximized:

$$\phi(x_1, x_2, \cdots, x_n) = \max = \phi(\bar{x}_1', \bar{x}_2', \cdots, \bar{x}_n') \quad \Big\| \quad \phi(q_1, q_2, \cdots, q_n) = \max = \phi_1^{\alpha} \equiv \phi(q_{11}^{\alpha}, q_{12}^{\alpha}, \cdots, q_{1n}^{\alpha})$$

at

$$b_1x_1 + b_2x_2 + \cdots + b_nx_n = L_2. \quad \Big\| \quad \sum p_1 q = {}_1R_0{}^0.$$

(It is necessary to bear in mind that

$$L_2 = \sum b_i\bar{x}_i = \sum b_i\bar{x}_i'.) \quad \Big\| \quad {}_1R_0{}^0 = \sum p_1q_0{}^0 = \sum p_1q_1{}^{\alpha}.)$$

Had consumption at the new prices $(b_1, b_2, \cdots, b_n)$ remained as before (equal to $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$), then the same standard of living would have continued in effect (the first postulate), but it would not have been a maximum one, for it becomes a maximum one only when new quantities of goods $(\bar{x}_1', \bar{x}_2', \cdots, \bar{x}_n')$ are consumed. Therefore

$$(4) \qquad \phi(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) < \phi(\bar{x}_1', \bar{x}_2', \cdots, \bar{x}_n'), \quad \Big\| \quad \phi_0{}^0 < \phi_1{}^{\alpha},$$

which was to be proved.[6]

3. INEQUALITIES WHICH BOUND THE TRUE INDEX OF THE COST OF LIVING

In consequence of the theorem proved above, there arise inequalities which determine the value of the true index of the cost of living.

$$(2'') \qquad \frac{\frac{\partial\phi}{\partial x_1}}{b_1} = \frac{\frac{\partial\phi}{\partial x_2}}{b_2} = \cdots = \frac{\frac{\partial\phi}{\partial x_n}}{b_n}. \quad \Big\| \quad \frac{\phi_{q_1}}{p_{11}} = \frac{\phi_{q_2}}{p_{12}} = \cdots = \frac{\phi_{q_n}}{p_{1n}}.$$

The corresponding numerators would have been the same, and therefore the corresponding denominators must be proportionate:

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}. \quad \Big\| \quad \frac{p_{01}}{p_{11}} = \frac{p_{02}}{p_{12}} = \cdots = \frac{p_{0n}}{p_{1n}}.$$

That is, consumption of *all goods* does not change except when all prices simultaneously rise or fall in the same proportion. The preceding conclusion is of course correct only when, in the equations (2′) and (2″), no numerator becomes zero; the conditions

$$\frac{\partial\phi}{\partial x_1} = 0, \ \frac{\partial\phi}{\partial x_2} = 0, \cdots, \frac{\partial\phi}{\partial x_n} = 0 \quad \Big\| \quad \phi_{q_1} = 0, \ \phi_{q_2} = 0, \cdots, \phi_{q_n} = 0$$

would have meant that the function ϕ at these values had an absolute maximum. We exclude from consideration in what follows such values $(x_1, x_2, \cdots, x_n)$ as correspond to a complete satiation of all wants.

[6] From the preceding footnote it is clear that the theorem holds true for any system of values $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ with one exception only, namely, that system in which there occurs a complete satiation of all wants. In this latter case, the proved inequality becomes an equality.

We pointed out above that:

$$\bar{\phi} = f(a_1, a_2, \cdots, a_n, K_1). \quad \Big\| \quad \phi_0{}^0 = f(p_{01}, p_{02}, \cdots, p_{0n}, R_0{}^0).$$

Let us now make the natural assumption that the general status of want-satisfaction—the standard of living ($\bar{\phi}$)—rises or falls (within certain limits) concomitantly with the rise or fall of the total expenditure for all goods, prices remaining unchanged. Thus we find that the consumer, in order to attain the same standard of living he enjoyed in the first period (at prices $a_1, a_2, \cdots, a_n$, and at the total expenditure of K_1 roubles) need only spend a sum of money smaller than

$$L_2 = b_1\bar{x}_1 + b_2\bar{x}_2 + \cdots + b_n\bar{x}_n. \quad \Big\| \quad {}_1R_0{}^0 = \sum p_1q_0{}^0.$$

Consequently the quotient

$$\frac{L_2}{K_1} = \frac{b_1\bar{x}_1 + b_2\bar{x}_2 + \cdots + b_n\bar{x}_n}{a_1\bar{x}_1 + a_2\bar{x}_2 + \cdots + a_n\bar{x}_n} \quad \Big\| \quad \frac{{}_1R_0{}^0}{R_0{}^0} = \frac{\sum p_1q_0{}^0}{\sum p_0q_0{}^0}$$

is larger than the quotient of the cost of living on the same standard, i.e., larger than the true index of the cost of living (J_1):

$$(5) \qquad \frac{L_2}{K_1} = \frac{\sum b\bar{x}}{\sum a\bar{x}} > J_1 \quad \Big\| \quad \frac{{}_1R_0{}^0}{R_0{}^0} = \frac{\sum p_1q_0{}^0}{\sum p_0q_0{}^0} > I_{01}{}^0$$

or: *the true index of the cost of living on a standard determined by investigation of consumers in the base period is smaller than the budgetary index computed on the basis of quantities actually consumed in the base period.*

This proposition may be developed, and at the same time clarified, in the following manner: Suppose that a certain group of consumers receives money (for instance, wages) for consumption, in the so-called "merchandise roubles," i.e., according to the budgetary index. And in the process, in order to bring the budgetary index closer to reality, there occurs a periodic re-evaluation of the weights of the index, based on investigations of the budgets of these same consumers, i.e., the consumer is guaranteed for one period of time exactly the same collection of goods which he consumes during the preceding period. Then, in accordance with the proved theorem, the standard of living of these consumers will inevitably rise. An elementary example, selected at random, convinces us of this. Let us suppose that the average quantities consumed during a certain period were 40 lbs. bread (at 5 kopeks per lb.), 10 lbs. meat (at 20 kopeks per lb.), and 2 lbs. sugar (at 15 kopeks per lb.). The cost of the budgetary aggregate therefore equals 4 roubles, 30 kopeks. Assume that in another period the price of meat rises to 30 kopeks per lb. The consumer is given the cost of the previous collection at the new prices, i.e., 5 roubles, 30 kopeks. However, due to

the relative dearness of meat, the consumer will diminish his consumption of it somewhat, and will increase his consumption of bread and sugar; for instance, he will consume 7 lbs. meat, 52 lbs. bread, and 4 lbs. sugar. In a third period, let the price of meat decrease to 25 kopeks. The consumer will receive the cost of the collection of goods which he consumed in the second period, i.e., 7 lbs. meat at 25 kopeks, 52 lbs. bread at 5 kopeks, and 4 lbs. sugar at 15 kopeks, in all 4 roubles, 95 kopeks. For this money the consumer will be able to buy 40 lbs. bread, 10 lbs. meat, and 2 lbs. sugar, i.e., the very same quantities of goods which he consumed in the first period, and besides, he will have a balance of 15 kopeks which will permit him to increase his consumption of all the goods he uses. His standard of living will thus clearly increase. Further price movements, no matter in what direction they occur, *would invariably allow the consumer to raise his standard of living.*

We will find another limit to the true index of the cost of living if we apply the preceding conclusion to similar indexes, where the period of their computation (the second period, chronologically speaking) serves as the base. An index with a base in the "given" period we call "the purchasing power of the rouble." This is a quantity which is the reverse of the index usually computed (with a base in the first period, chronologically speaking); i.e., the usual index is equal to the reciprocal of "the purchasing power of the rouble."

Suppose that an investigation of certain consumers' budgets during the period of computing an index (the second period, chronologically speaking) shows an actual consumption $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$, at prices $b_1, b_2, \cdots, b_n$, and a total expenditure of K_2 roubles.

$$K_2 = \sum b\bar{y}. \qquad \Vert \qquad R_1{}^1 = \sum p_1 q_1{}^1.$$

(The standard of living determined by the consumption $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$, may differ from the standards which we have considered above, and which were determined from the consumption $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ and the consumption $\bar{x}_1', \bar{x}_2', \cdots, \bar{x}_n'$.) The cost of the collection $\bar{y}_1, \bar{y}_2, \cdots, \bar{y}_n$, of goods, in the base period (the first period, chronologically speaking) at prices $a_1, a_2, \cdots, a_n$, will be

$$L_1 = a_1\bar{y}_1 + a_2\bar{y}_2 + \cdots + a_n\bar{y}_n. \quad \Vert \qquad {}_0R_1{}^1 = \sum p_0 q_1{}^1.$$

So, if we reverse the problem, and take as the base of the index the period of its computation (the second period), and designate as J_2 the true index of the cost of living with its base in the first chronological period, and as J_2' the true index of the cost of living with its base in the second period (J_2' is thus "the purchasing power of the rouble"), then:

$$J_2' = \frac{1}{J_2}\cdot \qquad \Big\| \qquad I_{10}^1 = \frac{1}{I_{01}^1}\cdot$$

On the basis of inequality (5) we have:

$$\frac{L_1}{K_2} = \frac{\sum a\bar{y}}{\sum b\bar{y}} > J_2'. \qquad \Big\| \qquad \frac{{}_0R_1^1}{R_1^1} = \frac{\sum p_0q_1^1}{\sum p_1q_1^1} > I_{10}^1.$$

Consequently:

$$\frac{L_1}{K_2} > \frac{1}{J_2} \qquad \Big\| \qquad \frac{{}_0R_1^1}{R_1^1} > \frac{1}{I_{01}^1}$$

and hence

$$(6) \qquad \frac{K_2}{L_1} = \frac{\sum b\bar{y}}{\sum a\bar{y}} < J_2 \qquad \Big\| \qquad \frac{R_1^1}{{}_0R_1^1} = \frac{\sum p_1q_1^1}{\sum p_0q_1^1} < I_{01}^1$$

or: *the true index of the cost of living at a standard determined by an investigation of consumers in the period of the computation of the index is larger than the budgetary index computed on the basis of the quantities actually consumed in the period of the computation of the index.*

Since the standards of living of consumers, investigated in the base period and in the given period, may differ from each other, so the true indexes for the corresponding standards may also differ from each other, i.e.,

$$J_1 \gtrless J_2. \qquad \Big\| \qquad I_{01}^0 \gtrless I_{01}^1.$$

Similarly, the corresponding budgetary indexes may also, generally speaking,[7] be different from each other:

$$\frac{L_2}{K_1} \gtrless \frac{K_2}{L_1}. \qquad \Big\| \qquad \frac{{}_1R_0^0}{R_0^0} \gtrless \frac{R_1^1}{{}_0R_1^1}.$$

Therefore each of the true indexes of the cost of living (J_1 and J_2) may not be included within the limits set by the two budgetary indexes

$$\frac{L_2}{K_1} \text{ and } \frac{K_2}{L_1}. \qquad \Big\| \qquad \frac{{}_1R_0^0}{R_0^0} \text{ and } \frac{R_1^1}{{}_0R_1^1}\cdot$$

It is not difficult to prove that *between the standard of living of consumers of the base period and the standard of living of consumers of the given period there always exists some standard, for which the true index of the*

[7] Budgetary indexes in a particular case may be found equal when true indexes are unequal to one another. When true indexes are equal, the budgetary indexes are always unequal.

cost of living falls between the budgetary index computed on the basis of quantities consumed during the base period and the budgetary index computed on the basis of quantities consumed during the given period.

Indeed, it is possible to represent the true index of the cost of living at any given time as a continuous and single-valued function of the standard of living (which apparently arises from the condition that the prices of the base period and those of the given period are constant). For example, an index of the standard of living of the poor strata of the population may be relatively low; an index of moderately well-off strata, higher; and an index of rich people, highest, or vice versa, and so on; and at the same time, to each standard there corresponds but one true index of the cost of living. A proposition exists that, given two values of the independent variable of any continuous and single-valued function, we may always find between them at least one other value of the independent variables, at which the function will assume any intermediate value between the two values originally given. In accordance with this proposition, it is always possible to find, between the standards of living of consumers investigated in the base period and the given period, some standard of living whose true index (J) lies between the true indexes for the standards of living of consumers of the two periods (J_1 and J_2), i.e.,

$$J_1 \lesseqgtr J \lesseqgtr J_2. \qquad \Big\| \qquad I_{01}^{0} \lesseqgtr I_{01}^{e} \lesseqgtr I_{01}^{1}.$$

Comparing the two inequalities

$$J_1 \lesseqgtr J_2 \text{ and } \frac{L_2}{K_1} \lesseqgtr \frac{K_2}{L_1} \qquad \Big\| \qquad I_{01}^{0} \lesseqgtr I_{01}^{1} \text{ and } \frac{{}_1R_0^{\,0}}{R_0^{\,0}} \lesseqgtr \frac{R_1^{\,1}}{{}_0R_1^{\,1}}$$

and also, bearing in mind inequalities (5) and (6):

$$\frac{L_2}{K_1} > J_1 \text{ and } \frac{K_2}{L_1} < J_2, \qquad \Big\| \qquad \frac{{}_1R_0^{\,0}}{R_0^{\,0}} > I_{01}^{0} \text{ and } \frac{R_1^{\,1}}{{}_0R_1^{\,1}} < I_{01}^{1},$$

we will find that only six combinations are possible, as follows:

$$\frac{L_2}{K_1} > J_1 > J_2 > \frac{K_2}{L_1}, \qquad \Big\| \qquad \frac{{}_1R_0^{\,0}}{R_0^{\,0}} > I_{01}^{0} > I_{01}^{1} > \frac{R_1^{\,1}}{{}_0R_1^{\,1}},$$

$$\frac{L_2}{K_1} > J_2 > J_1 > \frac{K_2}{L_1}, \qquad \Big\| \qquad \frac{{}_1R_0^{\,0}}{R_0^{\,0}} > I_{01}^{1} > I_{01}^{0} > \frac{R_1^{\,1}}{{}_0R_1^{\,1}},$$

$$\frac{L_2}{K_1} > J_2 > \frac{K_2}{L_1} > J_1, \qquad \Big\| \qquad \frac{{}_1R_0^{\,0}}{R_0^{\,0}} > I_{01}^{1} > \frac{R_1^{\,1}}{{}_0R_1^{\,1}} > I_{01}^{0},$$

$$J_2 > \frac{L_2}{K_1} > J_1 > \frac{K_2}{L_1}, \quad \Bigg\| \quad I_{01}{}^1 > \frac{{}_1R_0{}^0}{R_0{}^0} > I_{01}{}^0 > \frac{R_1{}^1}{{}_0R_1{}^1},$$

$$J_2 > \frac{L_2}{K_1} > \frac{K_2}{L_1} > J_1, \quad \Bigg\| \quad I_{01}{}^1 > \frac{{}_1R_0{}^0}{R_0{}^0} > \frac{R_1{}^1}{{}_0R_1{}^1} > I_{01}{}^0,$$

$$J_1 < \frac{L_2}{K_1} < \frac{K_2}{L_1} < J_2. \quad \Bigg\| \quad I_{01}{}^0 < \frac{{}_1R_0{}^0}{R_0{}^0} < \frac{R_1{}^1}{{}_0R_1{}^1} < I_{01}{}^1.$$

As we see, J may also be selected so as to fall not only between J_1 and J_2, but also between

$$\frac{L_2}{K_1} \text{ and } \frac{K_2}{L_1}, \text{ i.e.,} \quad \Bigg\| \quad \frac{{}_1R_0{}^0}{R_0{}^0} \text{ and } \frac{R_1{}^1}{{}_0R_1{}^1}, \text{ i.e.,}$$

$$(7) \qquad \frac{L_2}{K_1} \gtrless J \gtrless \frac{K_2}{L_1}. \quad \Bigg\| \quad \frac{{}_1R_0{}^0}{R_0{}^0} \gtrless I_{01}{}^* \gtrless \frac{R_1{}^1}{{}_0R_1{}^1}.$$

4. THEOREM CONCERNING THE CONDITION OF APPROXIMATE EQUALITY OF STANDARDS OF LIVING

Suppose that we have the data from two investigations of consumption, these data being subdivided into groups according to the quantity of total expenditure of consumers (or in other words according to the height of their standard of living). Let us set ourselves the task of finding, in both periods under consideration, consumers having approximately the same standard of living. If the standards of living of any two consumers are approximately equal during both periods of time, it is evidently necessary that the relation between the costs of living of these two consumers should be close to the true index of the cost of living. Let us find, therefore, the condition which must be satisfied by the quantities of consumers' goods consumed by the consumers investigated in two different periods, so that the relation of the costs of living of these two periods should fall within the same limits as does the true index of the cost of living.

Let us adopt the following notation:

Quantities of goods consumed in the base (first) period:

$$x_1, x_2, \cdots, x_n. \quad \Big\| \quad q_{01}{}^0, q_{02}{}^0, \cdots, q_{0n}{}^0.$$

Prices of the first period, correspondingly:

$$a_1, a_2, \cdots, a_n. \quad \Big\| \quad p_{01}, p_{02}, \cdots, p_{0n}.$$

Quantities of goods consumed in the given period (second period):

$$y_1, y_2, \cdots, y_n. \quad \Big\| \quad q_{11}{}^1, q_{12}{}^1, \cdots, q_{1n}{}^1.$$

Prices in the second period:

$$b_1, b_2, \cdots, b_n. \quad \Big\| \quad p_{11}, p_{12}, \cdots, p_{1n}.$$

Cost of living in the first period:

$$K_1 = \sum ax. \quad \Big\| \quad R_0{}^0 = \sum p_0q_0{}^0.$$

Cost of living in the second period:

$$K_2 = \sum by. \quad \Big\| \quad R_1{}^1 = \sum p_1q_1{}^1.$$

Cost of goods consumed in the first period at the prices of the second:

$$L_2 = \sum bx. \quad \Big\| \quad {}_1R_0{}^0 = \sum p_1q_0{}^0.$$

Cost of goods consumed in the second period at prices of the first:

$$L_1 = \sum ay. \quad \Big\| \quad {}_0R_1{}^1 = \sum p_0q_1{}^1.$$

The budgetary index computed according to the weights of the base period:

$$\frac{L_2}{K_1} = \frac{\sum bx}{\sum ax}. \quad \Bigg\| \quad \frac{{}_1R_0{}^0}{R_0{}^0} = \frac{\sum p_1q_0{}^0}{\sum p_0q_0{}^0}.$$

The budgetary index computed according to weights of the given period:

$$\frac{K_2}{L_1} = \frac{\sum by}{\sum ay}. \quad \Bigg\| \quad \frac{R_1{}^1}{{}_0R_1{}^1} = \frac{\sum p_1q_1{}^1}{\sum p_0q_1{}^1}.$$

The true index of the cost of living for some standard between the standards of living of consumers of the two periods is $J[=I_{01}{}^{\epsilon}]$. It is known that

$$\frac{\sum bx}{\sum ax} \gtrless J \gtrless \frac{\sum by}{\sum ay}. \quad \Bigg\| \quad \frac{{}_1R_0{}^0}{R_0{}^0} \gtrless I_{01^{\epsilon}} \gtrless \frac{R_1{}^1}{{}_0R_1{}^1}.$$

Let us examine the following relations:[8]

$$\frac{\sum by}{\sum ax} \text{ and } \frac{\sum bx}{\sum ay} \quad \Bigg\| \quad \frac{R_1{}^1}{R_0{}^0} \text{ and } \frac{{}_1R_0{}^0}{{}_0R_1{}^1}$$

and let us prove that, if we have the equality

$$(8) \quad \frac{\sum by}{\sum ax} = \frac{\sum bx}{\sum ay}, \quad \Bigg\| \quad \frac{R_1{}^1}{R_0{}^0} = \frac{{}_1R_0{}^0}{{}_0R_1{}^1},$$

[8] N. S. Chetverikov drew my attention to the possibility of such a formulation of the problem.

then

$$\frac{\sum bx}{\sum ax} \gtrless \frac{\sum by}{\sum ax} \gtrless \frac{\sum by}{\sum ay}, \qquad \Big\| \qquad \frac{{}_1R_0{}^0}{R_0{}^0} \gtrless \frac{R_1{}^1}{R_0{}^0} \gtrless \frac{R_1{}^1}{{}_0R_1{}^1},$$

i.e., *if the cost of living of consumers investigated in the given (second) period is related to the cost of living of the base (first) period as is the cost of goods consumed in the first period, evaluated at the prices of the second period, to the cost of goods consumed in the second period evaluated at the prices of the first period, then this relation falls between the same limits as does the true index of the cost of living for some standard intermediate between the standards of living of consumers of the first and second periods; and consequently, the standards of living of the two periods are approximately equal.*

Let us make two possible hypotheses. First let us suppose that

$$\frac{\sum bx}{\sum ax} > \frac{\sum by}{\sum ax}. \qquad \Big\| \qquad \frac{{}_1R_0{}^0}{R_0{}^0} > \frac{R_1{}^1}{R_0{}^0}.$$

Since the denominators are equal, then

$$\sum bx > \sum by, \qquad \| \qquad {}_1R_0{}^0 > R_1{}^1,$$

and therefore

$$\frac{\sum bx}{\sum ay} > \frac{\sum by}{\sum ay}, \qquad \Big\| \qquad \frac{{}_1R_0{}^0}{{}_0R_1{}^1} > \frac{R_1{}^1}{{}_0R_1{}^1},$$

or

$$\frac{\sum bx}{\sum ax} > \frac{\sum by}{\sum ax} > \frac{\sum by}{\sum ay}. \qquad \Big\| \qquad \frac{{}_1R_0{}^0}{R_0{}^0} > \frac{R_1{}^1}{R_0{}^0} > \frac{R_1{}^1}{{}_0R_1{}^1}.$$

Let us suppose the opposite:

$$\frac{\sum bx}{\sum ax} < \frac{\sum by}{\sum ax}. \qquad \Big\| \qquad \frac{{}_1R_0{}^0}{R_0{}^0} < \frac{R_1{}^1}{R_0{}^0}.$$

Then

$$\sum bx < \sum by; \qquad \| \qquad {}_1R_0{}^0 < R_1{}^1;$$

consequently

$$\frac{\sum bx}{\sum ax} < \frac{\sum by}{\sum ax}, \qquad \Big\| \qquad \frac{{}_1R_0{}^0}{R_0{}^0} < \frac{R_1{}^1}{R_0{}^0},$$

and

$$\frac{\sum bx}{\sum ax} < \frac{\sum by}{\sum ax} < \frac{\sum by}{\sum ay}. \qquad \Big\| \qquad \frac{{}_1R_0{}^0}{R_0{}^0} < \frac{R_1{}^1}{R_0{}^0} < \frac{R_1{}^1}{{}_0R_1{}^1}.$$

On the basis of the two hypotheses made above, we have

$$(9)\quad \frac{\sum bx}{\sum ax} \gtrless \frac{\sum by}{\sum ax} \gtrless \frac{\sum by}{\sum ay} \qquad \Bigg\| \qquad \frac{{}_1R_0{}^0}{R_0{}^0} \gtrless \frac{R_1{}^1}{R_0{}^0} \gtrless \frac{R_1{}^1}{{}_0R_1{}^1}$$

which was to be proved.[9]

5. A METHOD OF DETERMINING THE WEIGHTS TO BE USED IN COMPUTING THE TRUE INDEX OF THE COST OF LIVING

It follows from the above, that neither the budgetary index computed on the basis of consumption of the base period nor that computed on the basis of consumption of the given period can define the true change in the ccst of living at a given standard; and, moreover, one of them is greater and the other smaller than the true index. In order to construct the true index of the cost of living at a given standard, we must compute the price changes corresponding to changing quantities of goods consumed (changing because of the price changes). Hence we arrive at the necessity of establishing a relationship between consumption and prices, and of answering the question: What quantities should be consumed to guarantee a constant standard of living despite a given change in prices? The contents of this section will be made up of one of the methods of answering the question.

First of all let us note that, with the aid of equation (8) we can point out groups of consumers which have, in various periods of time, approximately equal standards of living. For this purpose we must have consumers' budgets, gathered through statistical investigations in each period, and these budgets should be grouped according to the quantity of the total expenditure for consumption of all consumers' goods. Then we must select from the two different periods those groups which satisfy equation (8). Thus, if we have consumers' budgets in the base period and in the given period, we can easily compute the true index of the cost of living at any standard by the budgets investigated in the two periods, by finding the ratio of the cost of living at identical standards.

[9] Equation (8) may be given an important meaning in the science of economics. With its aid there is the possibility, not only of finding the approximate value of the true index of the cost of living, but also of knowing approximately the changes in consumption which are dependent upon price changes. This latter question, consisting of the deduction and solution of a differential equation of the so-called "heterogeneity of indifference in consumption," we will postpone to a future number of *The Economic Bulletin*, and we will point out further only how to find variable (i.e., dependent upon price changes) weights for computing the index of the cost of living, with the aid of a system of a finite number of equations.

Let us consider those conditions which are sufficient to determine the quantities of goods consumed which correspond to a standard of living approximately identical with a given standard, at any changes in prices, without resorting to a new budgetary investigation each time.

We have the equation:

$$\frac{\sum by}{\sum ax} = \frac{\sum bx}{\sum ay} \quad \Bigg\| \quad \frac{\sum p_1 q_1^1}{\sum p_0 q_0^0} = \frac{\sum p_1 q_0^0}{\sum p_0 q_1^1}$$

or

$$(10) \quad \begin{aligned} &\frac{b_1y_1 + b_2y_2 + \cdots + b_ny_n}{a_1x_1 + a_2x_2 + \cdots + a_nx_n} \\ &= \frac{b_1x_1 + b_2x_2 + \cdots + b_nx_n}{a_1y_1 + a_2y_2 + \cdots + a_ny_n} . \end{aligned} \quad \Bigg\| \quad \begin{aligned} &\frac{p_{11}q_{11}^1 + p_{12}q_{12}^1 + \cdots + p_{1n}q_{1n}^1}{p_{01}q_{01}^0 + p_{02}q_{02}^0 + \cdots + p_{0n}q_{0n}^0} \\ &= \frac{p_{11}q_{01}^0 + p_{12}q_{02}^0 + \cdots + p_{1n}q_{0n}^0}{p_{01}q_{11}^1 + p_{02}q_{12}^1 + \cdots + p_{0n}q_{1n}^1} . \end{aligned}$$

In this equation let us consider as given the quantities of consumers' goods consumed during one period of time $x_1, x_2, \cdots, x_n$ and the prices $a_1, a_2, \cdots, a_n$ at which this consumption took place. The variables will be, first, quantities of goods consumed in some other period $y_1, y_2, \cdots, y_n$ which determine the same standard of living (the same general status of want-satisfaction of the consuming subject) as prevailed at the time of the consumption of goods in quantities $x_1, x_2, \cdots, x_n$; and secondly, the prices $b_1, b_2, \cdots, b_n$ at which the consumption $y_1, y_2, \cdots, y_n$ took place, will also be variables.

In order to compute the true index of the cost of living, which is approximately [given the condition that equation (8) is satisfied] the ratio $\sum by/\sum ax$, it is necessary to know the quantities $y_1, y_2, \cdots, y_n$ as functions of prices $b_1, b_2, \cdots, b_n$, i.e., at each price situation known to us $(b_1, b_2, \cdots, b_n)$ it is necessary to find the quantities of goods consumed $(y_1, y_2, \cdots, y_n)$ which define the same standard of living which existed at the given consumption $x_1, x_2, \cdots, x_n$ and at the prices $a_1, a_2, \cdots, a_n$. Equation (10) contains n quantities of goods consumed; consequently it contains n unknowns. For their determination we must have carried out an investigation of consumers' budgets at n different periods of time, at n different price situations, and with the material thus obtained we should have performed the operations described above: it would have been necessary to subdivide the budgets of each period according to the size of the expenditure for all consumers' goods, and to take for each period a group of consumers whose standard of living would have been equal to the basic standard existing in the first period, with the quantity of goods $x_1, x_2, \cdots, x_n$ consumed. These groups must be selected so that equation (8) would hold between the selected groups.

Suppose that the investigations of consumers' budgets mentioned above give us the following combinations of quantities of consumers' goods consumed as determining an identical standard of living:

In the first period, at prices $a_{11}, a_{12}, \cdots, a_{1n}$, consumption equalled $x_{11}, x_{12}, \cdots, x_{1n}$. (The first subscript accompanying x and a defines the period for which we take consumption and prices; the second subscript distinguishes one kind of consumers' good from another.)

In the second period, at prices $a_{21}, a_{22}, \cdots, a_{2n}$, consumption equalled $x_{21}, x_{22}, \cdots, x_{2n}$, and so on.

In the nth period, at prices $a_{n1}, a_{n2}, \cdots, a_{nn}$, consumption equalled $x_{n1}, x_{n2}, \cdots, x_{nn}$.

Between x and a of the first period and of all other periods the following relations should be maintained, as we have agreed:

$$\frac{a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1n}x_{1n}}{a_{21}x_{21} + a_{22}x_{22} + \cdots + a_{2n}x_{2n}} = \frac{a_{11}x_{21} + a_{12}x_{22} + \cdots + a_{1n}x_{2n}}{a_{21}x_{11} + a_{22}x_{12} + \cdots + a_{2n}x_{1n}}, \quad \Bigg\| \quad \frac{\sum p_0q_0^0}{\sum p_1q_1^0} = \frac{\sum p_0q_1^0}{\sum p_1q_0^0},$$

$$(11) \qquad \frac{a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1n}x_{1n}}{a_{31}x_{31} + a_{32}x_{32} + \cdots + a_{3n}x_{3n}} = \frac{a_{11}x_{31} + a_{12}x_{32} + \cdots + a_{1n}x_{3n}}{a_{31}x_{11} + a_{32}x_{12} + \cdots + a_{3n}x_{1n}}, \quad \Bigg\| \quad \frac{\sum p_0q_0^0}{\sum p_2q_2^0} = \frac{\sum p_0q_2^0}{\sum p_2q_0^0},$$

and so forth.[10]

$$\frac{a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1n}x_{1n}}{a_{n1}x_{n1} + a_{n2}x_{n2} + \cdots + a_{nn}x_{nn}} = \frac{a_{11}x_{n1} + a_{12}x_{n2} + \cdots + a_{1n}x_{nn}}{a_{n1}x_{11} + a_{n2}x_{12} + \cdots + a_{nn}x_{1n}}. \quad \Bigg\| \quad \frac{\sum p_0q_0^0}{\sum p_{n-1}q_{n-1}^0} = \frac{\sum p_0q_{n-1}^0}{\sum p_{n-1}q_0^0}.$$

[10] Besides, x and a of all periods satisfy similar equations among themselves; since the standards of living at all periods are approximately identical, we may write (with abbreviations, and leaving with x and a as subscripts only the ordinal number of the period):

$$\frac{\sum a_2x_2}{\sum a_3x_3} = \frac{\sum a_2x_3}{\sum a_3x_2}; \quad \frac{\sum a_2x_2}{\sum a_4x_4} = \frac{\sum a_2x_4}{\sum a_4x_2}; \cdots; \quad \frac{\sum a_2x_2}{\sum a_nx_n} = \frac{\sum a_2x_n}{\sum a_nx_2};$$

$$\frac{\sum a_3x_3}{\sum a_4x_4} = \frac{\sum a_3x_4}{\sum a_4x_3}; \quad \frac{\sum a_3x_3}{\sum a_5x_5} = \frac{\sum a_3x_5}{\sum a_5x_3}; \cdots; \quad \frac{\sum a_3x_3}{\sum a_nx_n} = \frac{\sum a_3x_n}{\sum a_nx_3};$$

and so on. Altogether $n(n-1)/2$ equations, including equations (11).

The existence of equations (11) allows us to write a system of n equations of type (10) in which b and y will be variables:

$$\frac{\dfrac{b_1y_1 + b_2y_2 + \cdots + b_ny_n}{a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1n}x_{1n}}}{} = \frac{b_1x_{11} + b_2x_{12} + \cdots + b_nx_{1n}}{a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n}, \quad \Bigg\| \quad \frac{\sum pq^0}{\sum p_0q_0^0} = \frac{\sum pq_0^0}{\sum p_0q^0},$$

$$(12) \quad \frac{b_1y_1 + b_2y_2 + \cdots + b_ny_n}{a_{21}x_{21} + a_{22}x_{22} + \cdots + a_{2n}x_{2n}} = \frac{b_1x_{21} + b_2x_{22} + \cdots + b_nx_{2n}}{a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n}, \quad \Bigg\| \quad \frac{\sum pq^0}{\sum p_1q_1^0} = \frac{\sum pq_1^0}{\sum p_1q^0},$$

and so forth:

$$\frac{b_1y_1 + b_2y_2 + \cdots + b_ny_n}{a_{n1}x_{n1} + a_{n2}x_{n2} + \cdots + a_{nn}x_n} = \frac{b_1x_{n1} + b_2x_{n2} + \cdots + b_nx_{nn}}{a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n}. \quad \Bigg\| \quad \frac{\sum pq^0}{\sum p_{n-1}q_{n-1}^0} = \frac{\sum pq_{n-1}^0}{\sum p_{n-1}q^0}.$$

Solution of this system for all y's gives us quantities of goods consumed, depending on any random combination of prices, which determine a standard of living approximately identical with n standards known to us. We thus obtain variable quantities (depending upon prices) used in computing the true index of the cost of living.

Applying the usual grouping of single commodities used in the computation of price indexes, we can reduce the number of equations necessary, and consequently can cut down the number of statistical investigations of budgets necessary. Indeed, the index of the cost of living, with a base in the first period, will be:

$$(13) \quad J = \frac{b_1y_1 + b_2y_2 + \cdots + b_ny_n}{a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n}. \quad \Big\| \quad I_{0t}^0 = \frac{\sum pq^0}{\sum p_0q_0^0}.$$

Subdivide the entire collection of consumers' goods consumed into groups, using as a criterion in grouping the synchronousness of the fluctuations of their prices (i.e., commodities should be subdivided into such groups that we may postulate that the prices of all goods within a group move together in proportion). We may define such groups by segregating consumers' goods with similar conditions of production and consumption. (For instance, one group will be made up of consumers' goods differing from one another by their grades only, etc.) The larger

the number of goods we can include in any group, the smaller the number of equations we shall require in order to determine the true index of the cost of living, but on the other hand the greater the error that will be made when we suppose that the prices of the individual goods within a group move together (i.e., change proportionately). With the help of a special statistical investigation of price movements for lengthy periods of time, it is possible to select groups so that the error mentioned above will not perceptibly influence the final result—the index of the cost of living.

Let us divide and multiply each term of the numerator (13) (each term being the total cost of each of the goods consumed in the given period, evaluated at the prices of this period), by the corresponding price of this commodity in the base period:

$$J = \frac{a_{11}y_1 \dfrac{b_1}{a_{11}} + a_{12}y_2 \dfrac{b_2}{a_{12}} + \cdots + a_{1n}y_n \dfrac{b_n}{a_{1n}}}{a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1n}x_{1n}}. \qquad \Bigg\| \qquad I_{0t}{}^0 = \frac{\sum p_0 q^0 \dfrac{p}{p_0}}{\sum p_0 q_0{}^0}.$$

In the numerator and the denominator let us combine the groups of goods having synchronous price fluctuations, and in the numerator let us carry outside the parentheses (for each group) the common (common because of the synchronousness of price fluctuations) relative changes of prices b/a (the so-called partial indexes). Let us assume that the number of the resulting groups is g. Let us sum up the costs of goods of each group in the numerator as well as in the denominator. Let us call these sums $A_{11}, A_{12}, \cdots, A_{1g}$, which represent in the denominator the total cost of the goods of each group in the base period at the prices of the base period. In the numerator let us call $Y_1, Y_2, \cdots, Y_g$ the total cost of the goods of each group in the given period, at the prices of the base period. Let us call the partial indexes of the corresponding groups

$$\frac{b_1}{a_{11}}, \frac{b_2}{a_{12}}, \cdots, \frac{b_g}{a_{1g}} \quad \Bigg\| \quad \frac{p_1}{p_{01}}, \frac{p_2}{p_{02}}, \cdots, \frac{p_g}{p_{0g}}$$

$$J = \frac{Y_1 \dfrac{b_1}{a_{11}} + Y_2 \dfrac{b_2}{a_{12}} + \cdots + Y_g \dfrac{b_g}{a_{1g}}}{A_{11} + A_{12} + \cdots + A_{1g}}.$$

Thus, in order to compute the true index of the cost of living, we need, not the quantities of each good consumed, but the total cost of separate groups of goods, at the prices of the base period. In accordance with this, we can reduce the system of equations (12), by transforming it in an analogous manner:

$$\frac{Y_1\dfrac{b_1}{a_{11}}+Y_2\dfrac{b_2}{a_{12}}+\cdots+Y_g\dfrac{b_g}{a_{1g}}}{A_{11}+A_{12}+\cdots+A_{1g}} = \frac{A_{11}\dfrac{b_1}{a_{11}}+A_{12}\dfrac{b_2}{a_{12}}+\cdots+A_{1g}\dfrac{b_g}{a_{1g}}}{Y_1+Y_2+\cdots+Y_g}, \qquad \Bigg\| \qquad \frac{\sum p_0q^0\dfrac{p}{p_0}}{\sum p_0q_0^0} = \frac{\sum p_0q_0^0\dfrac{p}{p_0}}{\sum p_0q^0},$$

$$\frac{Y_1\dfrac{b_1}{a_{11}}+Y_2\dfrac{b_2}{a_{12}}+\cdots+Y_g\dfrac{b_g}{a_{1g}}}{A_{21}\dfrac{a_{21}}{a_{11}}+A_{22}\dfrac{a_{22}}{a_{12}}+\cdots+A_{2g}\dfrac{a_{2g}}{a_{1g}}} = \frac{A_{21}\dfrac{b_1}{a_{11}}+A_{22}\dfrac{b_2}{a_{12}}+\cdots+A_{2g}\dfrac{b_g}{a_{1g}}}{Y_1\dfrac{a_{21}}{a_{11}}+Y_2\dfrac{a_{22}}{a_{12}}+\cdots+Y_g\dfrac{a_{2g}}{a_{1g}}}, \qquad \Bigg\| \qquad \frac{\sum p_0q^0\dfrac{p}{p_0}}{\sum p_0q_1^0\dfrac{p_1}{p_0}} = \frac{\sum p_0q_1^0\dfrac{p}{p_0}}{\sum p_0q^0\dfrac{p_1}{p_0}}, \tag{14}$$

and so on:

$$\frac{Y_1\dfrac{b_1}{a_{11}}+Y_2\dfrac{b_2}{a_{12}}+\cdots+Y_g\dfrac{b_g}{a_{1g}}}{A_{g1}\dfrac{a_{g1}}{a_{11}}+A_{g2}\dfrac{a_{g2}}{a_{12}}+\cdots+A_{gg}\dfrac{a_{gg}}{a_{1g}}} = \frac{A_{g1}\dfrac{b_1}{a_{11}}+A_{g2}\dfrac{b_2}{a_{12}}+\cdots+A_{gg}\dfrac{b_g}{a_{1g}}}{Y_1\dfrac{a_{g1}}{a_{11}}+Y_2\dfrac{a_{g2}}{a_{12}}+\cdots+Y_g\dfrac{a_{gg}}{a_{1g}}}. \qquad \Bigg\| \qquad \frac{\sum p_0q^0\dfrac{p}{p_0}}{\sum p_0q_{g-1}^0\dfrac{p_{g-1}}{p_0}} = \frac{\sum p_0q_{g-1}^0\dfrac{p}{p_0}}{\sum p_0q^0\dfrac{p_{g-1}}{p_0}}.$$

The number of equations (14) equals the number of group indexes. Solution of this system of equations gives the weights for computing Y true indexes of the cost of living, as the function of indexes of separate groups of goods b/a_1.

The reduction of the number of the groups of goods having similar fluctuations of prices can be carried on only up to a certain point (for instance, up to 30–40 groups); therefore, computation of the true index of the cost of living, while using a system with a finite number of equations, is connected with extraordinary difficulties (30–40 preliminary budgetary investigations!).[h]

[h] At the request of Professor Konüs the last sentence which relates to a forthcoming publication has been omitted.—H. S.

CONTROLLING THE NATION'S BUSINESS

By Edward Adams Richardson

I. INTRODUCTION

The establishment of the N.R.A. was a development considered favorable at the time, since it aimed at business control by Business, but proved a mistake when business men failed to understand what was required of them; the result was failure through adoption of the worst policies. It is not important for our purpose to discuss many of the other developments, though some of them will enter in later.

In this paper, Money will bulk large in the discussion. I shall have occasion to refer to the works of John Maynard Keynes, particularly *A Treatise on Money* and *The General Theory of Employment, Interest and Money,* even though neither is fully satisfactory. A sympathetic point of view will be found in the work of Dr. Douglas of the University of Chicago. Dr. Elmer Bratt of Lehigh University has done good work in gathering material together in his *Economics of Change.* Those much better informed than I with regard to what has been done will see herein the ideas of other men who may or may not have helped in the development of my point of view.

Whereas Keynes concerned himself with banking control in the *Treatise,* and with business in the *Theory,* I shall confine myself largely with the Community as a Whole, or Nation, and go back from there to the individual person or business. Though making Money serve as as the guiding concept, it is not banking control but rather essential co-operation in business management which will be developed.

II. DEFINITIONS

A number of definitions are in order. Some of them will be taken full-fashioned from the works of others, but many must stand by themselves.

Money: The measure of production and consumption, income and earnings. The base or form is of no particular consequence to us.

Income: For the individual, money received during a stated period for goods or for services rendered. Among the sources of income are wages, salaries, fees, commissions, dividends, interest, royalties, gifts, and others. Where an individual, or group, operates a business, the dividends or shares from the business (net income) are meant; hence rent, and other forms of income, which cover expenses as well as reward for services, appear as dividends. Income, so defined, covers the "production" due to individual contribution and nothing else.

Business: A non-income-earning entity whose sole concern is car-

rying on production. A business manages the capital arising from individual or corporate investment, provides for the increase thereof out of earnings and the supply of the wastage thereof in production, absorbs goods and services of others in production, disposes of the product, and acquires certain monies thereby for retiring investment, for dividends to owners, and (temporarily) surpluses and reserves of various sorts.

NATIONAL INCOME: The sum, during an assigned period, of all money incomes received by all individuals in the country. This is not quite the same as the production of the country, for allowance must be made for goods and services received from abroad and furnished abroad. It is what we shall call Earnings and denote by E. (See also Business Savings, as part of E; transfers from Business to others.)

ASSIGNED PERIOD: Any limited period between two set times, during which our difference (not differential) equations are to be set up, and the various quantities accrue.

SALES VALUE: The dollar value of all goods and services "consumed" during the assigned period, for which payment is made in some accepted form, either by cash payment, check, note, or the like. Sales Value is measured only as goods and services are received by the ultimate consumer, if consumption goods, or by a business if certain capital goods. This quantity for the nation will be denoted as V. Allowance must be made for foreign trade in both directions. Goods and services for foreign delivery, though paid for with money, are not purchased out of Earnings by the nation's people, while imports are.

CAPITAL: Goods and services used up in production which do not appear in the finished product. A means for increasing the efficiency of production. This definition is somewhat special. It does not cover Stocks of finished or unfinished goods.

INVESTMENT: Money utilized by others than the earners thereof. Such money may be used in increasing stocks or in the purchase of capital or in increasing individual consumption. Also, the algebraic sum, for the community, of all acts of investment or disinvestment (borrowings made, or borrowings repaid). This is the *change* in Investment Total and may be either positive (increase) or negative (decrease.) Denoted by I.

SAVINGS TOTAL: For the community, the total of monies, not spent by individuals who earned them, standing to their account, irrespective of the period in which earned.

INVESTMENT TOTAL: For the community, the total of monies on loan to others than those who earned them, irrespective of the time borrowed. Money on deposit, or the like, is not investment; when put to use by borrowers for purchasing, it is.

Both Total Savings and Total Investment will be corrected for losses. A lost saving becomes a "purchase" at the time the loss is acknowledged.

Savings: That portion of Earnings or Income which the earners of the community apply to increase (or decrease) the Savings Total of the community. An increment of income which is not spent during the assigned period is positive as an increment to national Savings; an increase of income by a withdrawal of some of his own savings is a negative increment to national Savings. The sum total of these positive or negative Savings increments may yield either positive or negative Savings. Savings is denoted by S.

Profit, Gross: This term has its usual accounting significance.

Profit, Net: This term, likewise, takes its usual meaning. It is the customary measure of the income from a business.

Profit, Earned: This term is similar to that of Keynes in the *Treatise;* it is the residue of Net Profit after allowance has been made for changes in the value of money, or windfall gains and loans for consumption. It should be noted that all earned profit, for a business, which is not paid out to individuals, may be credited to individuals on the one hand, and to Savings on the other.

III. THE WORKING OF THE MECHANISM

With these definitions in mind, certain equations will be established. Their utility will then be discussed. We desire to balance Sales Value (V) in terms of Earnings (E).

$(E-S)$ is the amount of money spent on consumption by individuals during the period. It is possible, of course, that a portion of Investment (I) was also spent by individuals.

I is the total amount of investment made, in part, from Savings.

If the definitions have been set up correctly, then

$$(E - S) + I = E + (I - S) = V \text{ (completed sales).}$$

The case of individuals is simple; that of Business is somewhat more complicated. This has been discussed by Keynes in the *Theory.* In a business, stocks of finished and unfinished goods may vary; capital is in part built up, in part used up; these things represent investment. The net positive investment might be calculated, business by business, and the sum for the community might be added to Investment. But such actions must be directly paralleled by corresponding acts of saving. For each positive saving there must be a positive investment, and vice versa. Nothing is gained by increasing Savings and Investment for the community by equal amounts. However, any money residue for the assigned period which is not paid out in dividends, or

invested in the business itself, but is put in the bank as surplus or reserves, should be accounted both to Earnings and to Savings. Similarly any bank borrowings or security issues should be accounted to Investment. The business "person" may not have an income, but it can let others use excess funds, and it does use the excess funds of others.

If Keynes in his *Theory* were correct, $(I-S)$ would always be zero. But to have this true, money deposited in a teapot by a bank-shy person would represent an increment of Savings, while the money itself would be an increment of Investment. This does not accord with my definitions. It is one thing to represent value, another to have intrinsic value as the result of productive action. The money is not equivalent to a purchase. Similarly, it is quite possible to deposit money in a bank, yet it may be difficult or impossible for the bank to loan it. "I" can, and sometimes does, differ appreciably from S. Increases in loanable money by credit creation can increase I without a corresponding increase in S. If these things are true, then I can exceed S through the absorption of that portion of Total Savings hitherto not invested. E, I, and S are "differences" applying to the assigned period.

Neglecting small errors due to receipts of money but failure to deposit in savings, the equation as given actually represents the important business of consumption of the product of industry. It is well, therefore, to investigate briefly certain particular cases.

The simplest is that in which $I=S=(I-S)=0$. In that case, $E=V$. Prices are so adjusted that the goods produced may be fully disposed of with the money earned in their production. Business is just replacing the capital used up in production. The savings of one period are being spent out of those being made during this period. Some businesses may be investing in plant expansions, while others are winding up their affairs. There is nothing static about the equation excepting that it will be more commonly associated with a constant, presumably high, level of employment. Since $(I-S)$ is zero, business men do not have to correct their net profits to obtain an earned profit. This equation represents the operation of a Capitalistic system with all that that implies. Nothing is said about the distribution of earnings between the wage workers and the owners of business excepting that the Keynesian "Propensities to Consume" of the two classes have stabilized. So long as $E=V$, business can maintain operations with all employed; this refers to monetary effects, of course, and assumes flexibility in stopping the supply of goods no longer wanted and in meeting new wants.

The next most simple state of affairs is that which Keynes considers typical; and that is, $(I-S)=0$, but neither I nor S is zero. Keynes appears to believe that a progressive community must have a positive value of I to meet the needs of new undertakings. If such is the case,

then the "Propensity to Save" becomes most important, for too much saving may make $(I-S)$ less than zero and lead to selling below cost to dispose of the product, while too little saving makes $(I-S)$ positive and goods are sold above cost. It is not obvious that either I or S should depart from zero permanently, nor that oscillations in the value of $(I-S)$ should set up the business cycles according to the *Treatise*, though it is obvious that they may.

An assumption that I continues positive indefinitely must be based on a miscalculation of capital wastage. Capital has a finite life which seldom exceeds 50 years even after allowing for replacement of wastage. A properly managed business will adopt a reasonable policy for full retirement of the original investment within such period as will avoid undue chance of loss through wearing out or obsolescence. It may use such disinvestment along with savings taken from Net Profits to reinvest in the business, if the business is growing, or it may invest such funds in other growing businesses, or repay them to the bondholders or owners. A value of I always positive would seem to indicate that those who save are expected to take losses which are never declared. Or else that no distinction is made between dividends which are normal earnings, and those which represent disinvestment operations. A certain small growth in I is to be expected in a growing country, or one which is making heavy new investments, but failure to have a positive value for I should not operate to slow down or prevent normal business. Most new investment can be made up from disinvestment or business savings. We must look further for a satisfactory cause of business cycles. Yet, as a matter of fact, it seems that the 1929 depression arose with our second condition satisfied.

It is necessary to note that I consists of money loaned to individuals for consumption purposes, as well as that for the increase in stocks and the purchase of increasing amounts of capital. This inclusion is important. Instalment buying is a good thing in many ways. But stability of such purchase methods depends in no small measure upon the maintenance of the high level of business and also upon the increase in the total of Instalment Loans being slow. If that total remains nearly constant, the money received from those paying off their debts suffices to supply the demands of those acquiring new debts. Such a condition is consistent with a condition of I equal to zero; certainly the increment of I so employed is then substantially zero. But any large and sudden increase in Instalment Loans outstanding makes the I increment positive. There is certainly some limit to the proportion of income which can be outstanding. Unfortunately, if for any reason employment is caused to decrease, this increment of I becomes negative and may be most important in its effects upon the course of business.

The portion of I employed in the increase of stocks is also an unstabilizing influence; the increment in I may be large and positive as business builds up, but possibly even larger but negative when business starts to decline. The variations in the I increment for capital may not be so large, however.

Now a large value of S tends to force a building up of outlets for I; money seeks use that a profit may be made. It becomes of equal importance to see why S should vary, and some of the effects thereof. A few statistics show that a period of good business leads to large increases in the higher-bracket incomes, and considerable increases in those of the lower brackets. Statistics on savings show that the large number of small incomes save very little, either individually or in total, while the upper brackets make very large savings. The variations in the large incomes cause very heavy variations in savings with the condition of business. Keynes' original assumption of a fairly steady rate of savings does not apply in American practice.

Furthermore, a depression with unemployment soon dissipates any savings those in the lower income brackets may have acquired, while some time must elapse before the savings in the higher income groups are touched. A depression also reduces investment in new purchases of all sorts, but the short-time instalments are paid regularly, building up an excess of unused savings. Those savings which are dissipated, serve, of course, to maintain business at a level higher than could happen otherwise. Keynes has discussed some of these points quite thoroughly in his *Theory*.

Other sorts of unbalance arise when $I=S$. Instalment Loans are one sort of investment, Foreign Loans another, excess investments in business and government others, Brokers' Loans another. In order to appreciate the problem before us, it seems well to set down the very approximate table below which represents I for the period from about 1922 to 1929.

Reduction in National Debt	\$ 7,000 million	I Neg.
Increase in Instalment Loans	\$ 6,000 million	I Pos.
Increase in Broker's Loans	\$ 4,000 million	I Pos.
Increase in Foreign Loans	\$ 7,000 million	I Pos.
Increase in Investment (Abnormal)	\$20,000 million	I Pos.
Total of the above items	\$30,000 million	I Pos.

As a basis for reference, Earnings were about \$80,800 million during 1929, perhaps not over \$500,000 million for the period.

To maintain such a value of I, prices must have averaged 6 per cent in excess of cost if Savings were zero, or, if Savings were also \$30,000

million, then the prices were "correct." If the latter were the case, then $E = V$ and there was no change in the value of money. The question then becomes, Were the Investments normal or abnormal, and how did they arise?

Let us suppose, and the assumption is open to some question, that the Price Level for all goods and services was maintained. Let us also suppose, and such an assumption is justified, that the efficiency of production increased an important percentage, tending to increase V without any corresponding increase in the portion of Earnings received by those who would tend to buy rather than save. To be sure some progress in wage increases occurred, but much below the percentage increase in V. Either incomes in the higher brackets must have increased, or a new type of earner must have arisen. As a matter of fact, both occurred.

This period from 1922 to 1929 was the one devoted to increased sales effort. When one tries to sell goods for more than the money available for their purchase, increased sales effort becomes very necessary. In fact, increased sales are so important that salesmen able to secure results, or advertising campaigns of like character, or special literature of all sorts, become worth large expenditures. By making such large expenditures to the publicity and sales class of people, the money available in the community for buying goods was brought measurably close to that required to buy the goods. The sales campaigns succeeded, not because of the ability of the sales force to put out the goods, but because a part of Savings which might have accrued if the goods could be sold were invested in excess production costs. The increased efficiencies created by the engineer were largely dissipated in unnecessary distribution costs. It is to be understood, of course, that there is a certain necessary and optimum amount of sales effort required to place the product of industry. Modern sales methods as such are not subject to criticism, but merely the lavishing of money on relatively incompetent help and material which would not have been needed provided prices had been reduced or income to producers had been increased. Some of this criticism should apply to other inefficiencies in distribution such as excessive numbers of distributing outlets.

When the price level is being maintained, it becomes somewhat difficult to detect that prices are too high unless we note the essentially Consumption character of the type of loans offered as investments. Broker's Loans, Foreign Loans, Instalment Loans; all point to the excess of price above cost. Let us touch briefly upon them.

Broker's Loans in themselves are reasonably well safeguarded. But for every dollar of such loans, some one may have risked a dollar of his savings. When the crash comes, some $4,000 million of savings are

to be transferred from one group of persons to another; and those who received the high prices for stocks either spent the money represented, or have it in their own savings. It is the sudden transfer "in fact" of ownership of savings that so vitally affects the outlook on business, particularly when the shift comes in a period of a very few weeks.

Foreign Loans must always be open to suspicion. Foreign trade can and should be carried on through the exchange of goods and services, dollar for dollar. Any deviation from such a course deserves the most careful consideration. To be justified, a booming country requiring continued investment for several years may prove acceptable, but such action in investment should conform to a definite, well-understood policy. When such loans are required under conditions preventing other nations from selling sufficient goods and services to us to effect payment, owing either to dislocation or demoralization of foreign sales and producing organizations, or to high tariffs or other unfortunate commercial practices on our part, they are not justified; payment is doubtful and such loans tend to disrupt our own economy. Much could be written on this one topic alone. The large import of monetary metals in payment is just as undesirable as selling goods to others by loaning them the money to pay, for gold is of value only when used to buy useful goods and services. The effect of imports of gold is just the same as loaning money in equivalent amounts. The primary effects are quite undesirable; any secondary benefits of consequence may be secured at lower cost otherwise.

The question of abnormal Investment (Industrial and Governmental) really should be subdivided into the two general sections. The industrial section can be discussed better as soon as the question of Profits has been considered.

The Normal Profit of the *Treatise* was presumed to be one which could be retained by business men. These latter secure not only all Normal, but also all Windfall Profits. In view of the tabulation which I have furnished, the Normal Profit of Keynes may not be fully retainable when I and S differ from zero. It will be seen that not only the Windfall Profit $(I-S)$, but also the amount S may be unretainable by business men. When I differs from zero, the whole amount I becomes a factor which may serve to disrupt business. Although Keynes would find prices satisfactory when $(I-S)=0$, I find that, when I amounts to 6 per cent of income and much of I is used to finance consumption, prices are actually too high by at least the amount so used for consumptive purposes. This does not include the price rise, or rather failure to reduce prices, accounted for in the excess distribution costs which were built up primarily to maintain the price level. Small wonder, if such is the case, that the call for accounting in 1929

led to a long and deep depression. Something like $6,000 million in instalment loans had to be wiped out in less than two years, nearly $4,000 million in brokers' loans in a few months, foreign loans had to be materially curtailed, and capital invested with the expectation of a high rate of operations became immobilized. With the best of management and no unemployment, or wage cuts, a price drop of nearly 20 per cent for nearly one year became imperative, unless the rising Savings were to be utilized in new consumption in whole or in part. A voluntary cut of such magnitude was unthinkable. A depression forcing the necessary corrections, and such wastage as results from the resisting of such corrections, was the only way out provided by so-called "economic law" (of the jungle). Even such a price cut might not have been effective. Once psychology is given a chance to frighten the bewildered business men to death, it is already too late. A man may swallow a cow if given sufficient time, but may choke to death rushing the eating of a piece of chicken. So is it in our economic order; large changes may be effected gradually, but only disaster can follow sudden forced changes.

IV. SUGGESTION FOR A STABILIZATION METHOD

With the above common-sense principle in mind, let us see what can be done to avoid the monetary forces leading to unstable business cycles.

Suppose we can gather, say quarterly, data on the amount E for all individuals, the amount S for individuals and businesses (that for a business being excess funds loaned to others or placed on deposit), and the amount I for individuals and businesses (that for business being the borrowings from others). Then, provided suitable corrections have been made for foreign-trade effects, or such trade for our discussion is assumed zero, we know very closely the production of consumption goods, or V. Partial or quarterly returns for income-tax purposes, provided the returns are made in a form consistent with our definition (and the whole tax system should be made consistent) should enable us to gather these data. A check may be secured through universal social-security returns at quarterly intervals, together with other necessary data. If these figures are available, then we know I, S, E, and V. Each business man, further, can figure what he has added in production from the difference between gross sales and purchases, provided his accounting procedure is consistent with the definitions. The business man will know the ratio of I to V from published data. He can calculate the product of this ratio by the amount his business adds in production. This amount should be regarded as a Reserve.

Any positive value of I in any quarter suggests that *prices should be reduced on such items of production as can best stand the reduction for the*

next quarterly period. The aim is to secure a value of I differing little from zero, and to provide through Reserve for positive variations. Negative variations may be cared for by removing the necessary amount from Reserve. In fact, the purpose of such funds is to finance price lowerings necessary to care for variations in the values of I and S. Such reserves should come before dividends and surplus. Debt has one characteristic which is often misunderstood. Men find that "losses" must be taken in business. These losses, if taken, are nothing more than my Reserves. Two thousand and even more years ago people understood that debts were subject to miscalculation, so the rule of forgiving seven-year-old debts arose (Deuteronomy). There was much wisdom in the rule. Whether they should be made before taxes is another question.

If all business men can be induced or required to make such price adjustments as this theory requires, there will be no large building up of Investment which may serve to bring on disaster in times of crises. Under this system large changes are made regularly and gradually. The business men are not giving anything away; they are merely keeping books which tell them much more precisely how good their business really is by allowing for money which must be lost. They know how much in error business estimates were for the preceding quarter and can make correction for them at once. The Profits they secure are Earned and not Net. Such profits are the only ones which can tell whether to expand a business or not. No amount of inflation will distort them; the same applies to deflation.

These methods suggest valid undertakings for business organizations of the N.R.A. type and suggest why the N.R.A. failed as originally organized. Certain stages of production can take price reductions better than others; certain industries, or plants in industries, can take price reductions better than others; all of these things are matters for discussion and settlement by business men. In addition, such organizations can undertake sales "disarmament" and improve the efficiency of production. A plant working part time may be enabled to work full time, pay for the increased sales effort, and make a larger profit. But if some or all of the other plants in the industry do likewise, the net effect is merely a permanent increase in manufacturing costs (including sales cost in production) for the industry as a whole, and raising prices. It is to the advantage of all businesses to reduce sales effort to the optimum. The question is not simple, for some magazine advertising in part supports valuable and worth-while service to customers and others, some sales effort is likewise service worth the cost and more. The greatest economies can be secured by reducing the reward for sales "effort" by reducing the effort required.

Obviously the study of proper accounting methods, better methods of financing, distributor policies, and the like all enter in. The most important factor which can be furnished by business organizations is maintained "confidence." This question can be discussed a little later, as the relation of my suggestions to so-called psychology will be investigated.

The rapid gathering of basic data is a vital governmental function; it can be carried out in conjunction with tax gathering and the management of Social Security matters. There is room for co-operation between business and government through the Securities and Exchange Commission in devising suitable accounting methods and enforcing their application. In general, it will be found that the preferred relationship of government to business is that of co-ordinating body, not that of a directing one. Government does have the right and duty, however, of initiating corrective measures when business men refuse, or at least fail to do so themselves.

It may seem that business will always be setting up Reserves. Such is not the case, for there will be times when I is negative. At the bottom of a depression, surplus expended in making "losses" by permitting the sale of goods at a price low enough to stimulate buying, thereby forcing recovery, will soon be replaced as recovery proceeds. Though Keynes (*Theory*) appears to believe that price variations in changes in the rate of doing business are not important, actually they are of the greatest importance. When business has stabilized at a low level, it takes time and much investment in stocks of raw material and partially finished goods before a new and higher level of finished-goods production becomes stable. We place 3,000,000 men at work, most of whom receive pay weekly, thereby increasing E. It may take a year to get the new raw materials into finished products (as an average). Assuming $I=S=0$, and no great increase in the number of goods produced for a considerable period, V must increase in proportion to E, and prices similarly. Actually employment will occur by stages, and finished goods will follow a distinctive curve of production with time. Prices must be varied rather expertly with time in order to hasten and insure recovery. That this is so is well indicated by the number of hesitations and stoppages on the way up which have distinguished recovery in the United States. It is essential that the excess prices should be reduced rapidly as employment, and more particularly finished-goods production, approaches balance. During the recovery, the excess prices have forced saving and provided the necessary investment to finance such recovery. If prices are not lowered rapidly enough as equilibrium is approached, we have the strange sight of partial re-employment only occurring, and businesses making large profits such as

distinguished 1936 and 1937 in this country. Such excessive earnings before recovery is complete are not a favorable sign; rather they are a most ominous one and indicate extreme shortsightedness on the part of business men as a whole. Though such excess prices as are permissible on increasing rates of operation are not included in I or S, owing to our definitions, the excess should be credited to Reserve, for when the business suffers a decline, or liquidation, the excesses of the rise must finance the prices below cost of the decline, or the community will not have the wherewithal to absorb the goods. If no decline occurs, then this forced investment may remain in the business for a long period. This is a question for the individual.

A rough check on the behavior of business during depressions may be had by referring to the government statistics on income from wages and salaries, money from interest and dividends, and total income. Since habits change slowly, there will be a certain ratio of wages and salaries to national income which will correspond fairly well with stable business. When this ratio declines, a depression is in the making. As a matter of fact, the ratio declined towards 1929 and reached a very low (relative) figure, going even lower in 1930. By 1932 the ratio had gone above that for 1929 and business was reaching a state adapted to recovery. The ratio rose very slightly through 1935 but no figures have been available for 1936 and 1937. It would be better if the salaries were split into those which represent services and those which in part represent disguised dividends or partnership shares, but such is not the case. Ordinary salaries plus wages should be a much better guide to the behavior of business than all salaries plus wages. The low point in the ratio came in 1930 (perhaps determining the length and depth of the depression) owing to the increase in interest and dividends relative to wages and salaries. The "interest rate" was "sticky," about the worst possible thing for business as a whole. Instead of using the surpluses, which included my Reserves, for lowering prices by reducing interest, dividends, and the higher, dividend-carrying salaries, business men foolishly disbursed them, greatly increasing S temporarily without any corresponding I. Naturally maintained prices made goods unbuyable and forced a decline.

V. "PSYCHOLOGY"

Let us now turn our attention to the question of psychology. That term covers the excess optimism of booms and the melancholia of depressions. So far as Profit Motive and Reserves and their use is concerned, the proposals I have made greatly reduce the possible swings and probably avoid any general movement. The really important psychological effect is what I might term the Youth, or Sheep Effect.

A survey of the past two decades will show that the amount of, and the knowledge of business data have both increased. Many young men have studied the trends of business by scientific methods and have learned how to interpret trends and just what action to take when certain trends develop. Being much better informed than their elders, they have acquired increasing importance in industry. Unfortunately, the action of business has developed along the following lines: A thousand young men look at the charts one fine Monday morning and find that there has been a slight decline in index Q. According to instruction, it is imperative that certain steps be taken to discount the future of Q by reducing activity in the Q direction. A thousand businesses promptly take the necessary steps, each one of which adds its small increment to the further deflation of Q. Comes the next Monday morning. The thousand young men in their wisdom are much surprised to see that their worst fears have been realized, so a thousand businesses take another reef in Q. It doesn't take many such reefs to wreck a mighty fine and promising line of business activity. Of course it works on the side of increase likewise.

Now a thousand misguided business men of the old style would not have known of the necessity for going easy on Q so would have pushed ahead in their muddling way. Q might have gotten better or worse, but probably not enough of either to affect matters much.

On the other hand, we might secure a thousand business men with the knowledge of tendencies equivalent to that of the young men, who would read the Q trend and say to themselves. "If all of us contract, business will get worse; if I contract alone, I alone lose a profit for a time; if I assume that every one will be sensible and take steps to boost Q, and no one else follows me, I may make a real loss. Suppose we get in touch with the others and lay down a policy which we can be reasonably sure will be generally followed." A new use for the type of organization which the N.R.A. was supposed to be has arisen.

The old way of doing business was occasionally disastrous. The proposed way of doing business, carried out with good will and fairness, can be most effective in maintaining business. The present way, in which people know the worst too soon and act too quickly as "rugged"(?) individualists, is about the worst possible. It contributed to our great depression. The rugged individualist may know what he ought to do provided all others do likewise, but considers what he should do when he is reasonably sure many others will do likewise of their own accord. So all act in much the same way, each to protect himself as best he may, all actually running "sheep-like" in the same direction and towards the cliff.

Many problems of everyday business require co-operative assurance

of common action. Crises demand a plan of enforced common action. Hence any organization, whether of business men, business men and government, or by government, which can make the aims of the old N.R.A. effective and enforce the minimum of necessary common action, has a large place in any scheme of economic stabilization. Given time, the business men themselves will be encouraged to act for the general welfare (thereby forwarding their own) on the assumption that there will be enough like-minded men to warrant such behavior. It is much more important to insure that 95 per cent of the business capacity of the country will co-operate than to spend too much time on the 5 per cent who chisel.

VI. THE EARNINGS FACTOR

So far, discussion of control has been confined to Prices, Reserves, Profits, and Co-operation. It should be obvious that the E factor and its make-up is important and may contribute much to the best solution of the various problems involved. A few words are in order.

As many people as may desire to earn money in production should be enabled and encouraged to do so; retrained, if necessary, for the purpose. Not the monetary but the physical side governs. The Capital and Interest factors of Keynes are not final. The more people who work, the higher the standard of living for the community may become. There is no particular reason why old or young, men or women, free men or convicts should be discriminated against; on the contrary, more general participation in production, at wages in some measure corresponding to efficiency and ability, will reduce the heavy load of dependents and will make possible greater satisfaction for many. This is particularly true in the case of those past 40. Hindering the old in their search for employment is not only a social, but an economic crime. I have known of too many cases where the inefficiency of production has been increased through arbitrary substitution of men just because of age. Besides, young men are seldom veterans such as are required to "laugh off" what seem like crises. Reasonable attempts at personnel and job classification would soon show that all desiring work might be employed. The limit on consumption is set by my equation, not by the number employed. If insurance or pension considerations are operative, the former can be overcome by requiring group insurance rates to be based on the effective average age for the community, not that for the plant in question; the latter, by continuing the development of a suitably effective and inclusive Social Security policy.

The great problem of E is embodied in the serious discrepancies in wage rates sectionally, and by trades, for equal or equivalent performance requirements. Labor has a large problem to solve in adapting

itself to promoting its cause with respect to, and not contrary to, the general welfare. Arbitrary raises of rate, according to what the traffic can be forced to bear, are not politic and may prove quite harmful. Efforts at exclusion and limitation to maintain rates are understandable, but cannot be justified when monopoly prices, or prices adapted to secure a comfortable annual income for a short period of work, are the result. Greater efforts must be made to reduce such rates through smoothing out seasonal variations, and through two-job training.

Foreign-trade considerations enter in to determine the general level of wages and income. Inasmuch as wage rates are distinctly sticky, it is probable that corrections for wage levels getting out of line will rely upon variations in the value of the dollar, and not upon actual variations in wage rates *en masse*. The aim here is to avoid too sudden fluctuations in the course of foreign trade.

In particular what can be said of the Control when such affairs as wars must be fought? So much of economic depression has been blamed on the war. The obvious answer is simple; tax freely and avoid debt creation if possible. In such case, we split E into two parts, E_c going to the people for their needs, and E_t going into taxes. Some difficulty must be expected arising from the shift in the type of consumer demands and this will raise the cost of production much as would occur upon a change in the level of activity for the community as a whole. Excess prices to finance the shift are subject to the same rules as in the case of finished goods produced during such a production increase. Similar allowances must be made for reserves to meet the cost of scrapping such war plants, and the rate of depreciation must be high. When the war is over, investment is necessary to permit a return to normal activities. Obviously time will be required for new industries to grow up. Yet there will be many needs to be met in the nature of replacements of houses, furniture, clothing, and the host of things which the people have been too poor to buy during the war. Considerable wise planning is needed. Yet there is little need for adjustment otherwise. If taxation has been the weapon used, wages will not get very far out of line and will need little correction. There will be no debts and no interest on debts to be met. The needs of the community are so great that full production for several years should be possible, enough to give time for new products to develop.

When money is borrowed, it is usual to have credit created in part, and I becomes greater than S. Wages may seem larger and may be forced upwards more readily as I forces V to increase, but so far as physical welfare is concerned, there is no difference between borrowing money and taxing, excepting only that interest charges arise which have the effect of changing the distribution of income. Even then, a

proper taxing system will virtually absorb not only the interest but the principal, only the accounting being delayed. Failure to understand the necessary relations existing do operate to change the point of view and influence business judgment. The future, contrary to common beliefs, never pays for what is done in the present; any physical deprivation takes place here and now: why not be realistic and so account for it?

VII. SUMMARY

Let us briefly review some of the important points of the type of control which I have described:

1. An equation connecting Sales Value for the Nation with Income, Investment, and Earnings, subject to definitions.

2. A method for estimating the excess part of V to be held in Reserve to finance Price Lowerings.

3. A method for adjusting prices, business by business, period by period, to avoid accumulating credit unbalances which may force a depression.

4. A method, based on the isolation of such Reserves, for securing a profit measure corrected to eliminate funds temporarily held which must be lost.

5. It is to be particularly noted that the above methods have been devised to aid business men in adopting accounting procedures which will be consistent with Capitalistic theory. The necessary Price Flexibility is made possible and a reliable Profit Motive is secured.

6. The aim of business control is the maintenance of production at a continued high level, at least for the Nation. Corrective steps, regularly applied, insure maintained money demand without recessions.

7. Justifiable excess prices, and the setting up of reserves therefor, involve expansion of national production, or a transformation in the character of production as in passing from peacetime to wartime economy.

8. A continued high level of operation makes possible the adoption of policies for maintained rates of capital-goods production and replacement, subject at most to narrow variations in volume.

9. The last nine years represent the taking of a loss in physical production amounting to about, or more than, 150 billion dollars because of a 30-billion-dollar *bookkeeping* error which should have been corrected in full without a recession.

10. Government control, at least in the United States, may be very broad, but the actual central control required is primarily the gathering of basic data, the standardization of accounting methods relative to reserves for price corrections, and the enforcement of such

relatively simple procedures; always subject to the right to intervene when business men fail to take the necessary measures of their own accord. Centralized business control is neither necessary nor desirable.

11. Certain problems requiring the collaboration and co-operation of business men are suggested, primarily to remove the "Sheep Effect" in times of crisis. The assurance of common action is the requisite for sound and effective business policies. The purpose of such common action, as before stated, is to maintain operations.

12. The ratio of wages and ordinary salaries to National Income is a rough criterion of business condition. This idea involves a theory of interest regulation for the community.

13. Sales Costs in excess of the optimum (as measured at zero Reserve requirements) is an economic waste and an artificial method for increasing E to maintain a price level.

14. There is no sound basis to the assumption that Investment must be maintained positive and that capital production is dependent on Savings. Capital is a cost of production and can be financed without reference to savings.

15. Large changes may be made in business if made slowly enough. The corrections described are designed to avoid, or greatly minimize, crises, and do show how crises may be handled with little disturbance.

Bethlehem, Pennsylvania

A DYNAMIC SCHEME FOR THE BRITISH TRADE CYCLE, 1929–1937

By E. A. Radice

THE AIM of trade-cycle theories is to select certain variables which are considered to be fundamentally important in explaining economic phenomena and to discover how their variations give rise to cyclical movements. A theory may take many forms, depending on the nature of the variables originally selected for analysis, but must, as Tinbergen[1] suggests, be a "closed system" with as many equations as variables.

The object of this paper is to present a possible closed system which may be applied to the observed facts of the recent British cycle, and, by mathematical solution, to indicate the types of cyclical movement which may result from the original hypotheses. The hypotheses, which, mathematically speaking, take the form of equations connecting the variables must, it is clear, be neither too numerous nor too complicated, if determinate solutions are to be discovered. Following the suggestions of writers such as Frisch and Kalecki, we may restrict ourselves to hypotheses which take the form of linear equations between the variables or their first differentials, involving constants which are assumed to remain unchanged over the course of the cycle.

What relationships between important economic variables have been relatively constant over the past few years? This is a question of fundamental importance on which the whole analysis depends. It can be answered either by general economic reasoning, the most frequently adopted method, or by an examination of actual observed data. We shall start by using the second method, and gradually work back to the first.

In his *National Income and Outlay* (pp. 187, 255), Mr. Colin Clark shows that there has been an almost exact linear relation between business savings and that part of the national income which is made up of profits and interest. A similar, but less close, linear relationship can be shown to have existed between the savings performed by individuals and the incomes received by them.[2] These two empirically observed relationships are, in addition, acceptable on general economic grounds. The consumption habits of individuals do not change appreciably within the period of one trade cycle, and the hypothesis that a fixed proportion of any additions to their incomes is saved is therefore admissible. The simplicity of the first relationship is not so easily justified theoretically, since it might be thought that the rate of inter-

[1] J. Tinbergen, "Suggestions on Quantitative Business Cycle Theory," ECONOMETRICA, Vol. 3, July, 1935, p. 242.

[2] See Appendix at end of the paper.

est would be another variable closely affecting business savings. Nevertheless a simple linear relation between business savings and total profits and interest fits the observed facts so closely, that nothing would be gained by a more complicated hypothesis.

Two further hypotheses can be made which approximate closely to the observed facts. Firstly, depreciation may be assumed to be constant,[3] and secondly the savings of central and local governments may be assumed to be zero.[4] The first of these hypotheses is perhaps open to question on theoretical grounds, but is a very convenient simplification which accords well with the facts of this cycle, however dangerous its general use may be.

Let us now state our hypotheses in explicit form. The following notation will be used:

$E(t)$, a function of time, is the gross national income consisting of:

$E_1(t)$, the net income paid out to individuals,

$E_2(t)$, business savings,

ϵ, depreciation plus government savings (assumed constant);

$E_3(t)$ is total profits and interest;

$S(t)$ is the net savings of individuals; and

$J(t)$ is gross investment [net investment (=net savings) plus depreciation].

We have after differentiation:

$$E'(t) = E_1'(t) + E_2'(t), \tag{1}$$

$$J'(t) = E_2'(t) + S'(t), \tag{2}$$

$$S'(t) = \delta E_1'(t),\ \delta \text{ being a constant}, \tag{3}$$

$$E_2'(t) = \gamma E_3'(t),\ \gamma \text{ being a constant}. \tag{4}$$

Equations (4) and (3) are expressions of our first two hypotheses, and γ and δ have to be determined from the data.

The introduction of the variable gross investment (J) evidently requires the adoption of a further hypothesis regarding the relationship between investment and other economic variables. The assumption of a constant "multiplier" will not suit our purposes, since a multiplier is implicitly included in equations (3) and (4), so that any additional assumption regarding the multiplier would in fact only be a further hypothesis relating income to consumption. What we require is a relationship which includes one or more of the factors which influence the investment decisions of entrepreneurs. We might, for instance, assume that the level of investment is a function of current profits and the rate of interest. A linear function of this type fits the data

[3] See Clark, *op. cit.*, p. 86.

[4] This is approximately correct until 1937.

fairly closely, but no more closely than a simple linear relationship between total investment and total profits.[5] This may appear surprising to those theorists who regard the rate of interest as an important determinant of investment activity. No doubt a linear function whether of two or of three variables is inadequate to explain the relationship, and, if more complicated functions were to be used, the importance of the rate of interest might become more evident. Since our own scheme must remain relatively simple if the equations are to be manageable, we adopt a hypothesis which fits the data fairly well:

(5) $J'(t) = \alpha E_3'(t)$, α being a constant.

We now have five equations and six unknowns. So far none of the equations are of the differential type, nor have any time lags been introduced. Evidently our sixth equation will have to take a rather different form if the final scheme is to lead to a system containing cyclical variations. This sixth equation must, in fact, include certain "dynamic" elements which have so far been lacking.

Let us return to the question of what factors determine the investment decisions of entrepreneurs, paying particular attention to conditions in the market (we have already considered the influence of the current level of profits). Traditional theory, with its emphasis on free competition, has usually regarded movements of prices as the most important forces which influence the plans of individual entrepreneurs. More recently, the increased attention paid to monopolistic forces and to output as a whole (as opposed to output of individual firms), has led many theorists to regard changes in the level of total income rather than changes in prices as being the most important factors influencing total investment activity. This has led to the hypothesis that there is a simple relationship between net investment and the rate of change of current income.

If such a relationship exists in a simple linear form it is clear that a certain time lag must be introduced if both net investment and current income move cyclically and reach their peaks and troughs at about the same times. Thus, if they each followed sine curves, investment at any time would be related to the rate of change of income one-quarter of a cycle previously. However, there is no need to assume such regular cycles since each single cycle may have damping or anti-damping forces within it which may, for instance, be counteracted in the next cycle by appropriate changes in the various factors assumed constant within the first cycle. In any case the existence of such a lag seems to accord with the facts. It can perhaps be explained by the existence of

[5] See Appendix at end of the paper. The empirical data used are the figures given by Mr. Colin Clark.

unemployed resources at the trough of the cycle, and the continuance of the momentum of investment during a boom in excess of the needs of the growing volume of consumption. We make the assumption that this lag is constant throughout the cycle. Since net investment differs from gross investment by a constant term, we have, after differentiation,

$$J'(t) = \lambda E''(t-\theta),\ \lambda \text{ and } \theta \text{ being constants.} \tag{6}$$

We now have a closed system of six equations connecting the six variables E, E_1, E_2, E_3, S, J and five constants α, γ, δ, λ, θ, which must be determined from the data. If the solution leads to cyclical movements in $E(t)$ our system leads to a type of trade-cycle theory not unlike that of Mr. Harrod, who lays more stress on changes in variables such as income and savings than on variables such as the price level and the rate of interest.

Taking together equations (1), (4), and (2), (3), (4) we have:

$$\delta E'(t) = \delta E_1'(t) + \gamma\delta E_3'(t), \tag{7}$$

$$J'(t) = \delta E_1'(t) + \gamma E_3'(t), \tag{8}$$

$$\therefore\ J'(t) = \delta E'(t) + \gamma(1-\delta)E_3'(t), \tag{9}$$

$$\therefore\ \delta E'(t) = J'(t)[1-\gamma(1-\delta)/\alpha], \tag{10}$$

from (9) and (5).

Writing

$$\mu = \frac{1}{\delta}\left[1 - \frac{\gamma}{\alpha}(1-\delta)\right], \tag{10a}$$

and considering (6) and (10), we have

$$E'(t) = \lambda\mu E''(t-\theta). \tag{11}$$

Let:

$$E'(t) = Ae^{zt/\theta} \tag{12}$$

where A is a constant and z independent of t.

Then, from (11) we have:

$$e^z = z\lambda\mu/\theta. \tag{13}$$

The solution of this characteristic equation is fully discussed in the work of Frisch, Holme, James, and Belz.[6] If $0<\lambda\mu/\theta<e$, cyclical variations occur. Otherwise (13) has two real roots.

[6] Ragnar Frisch and Harald Holme, "The Characteristic Solution of a Mixed Difference and Differential Equation Occurring in Dynamic Economics," ECONOMETRICA, Vol. 3, April, 1935, pp. 225–239.

R. W. James and M. H. Belz, "On a Mixed Difference and Differential Equation," ECONOMETRICA, Vol. 4, April, 1936, pp. 157–160.

(11) is now satisfied by an equation of the form

$$E'(t) = Be^{x_1 t/\theta} \sin y_1 t/\theta, \tag{14}$$

where B is a constant, $x_1 + iy_1 = z_1$, and z_1 is a root of (13). The period n of (14) is given by

$$n = 2\pi\theta/y_1. \tag{15}$$

We have, inserting $x_i + iy_i$ for z in (13),

$$x_1 = y_1 \cot y_1, \tag{16}$$

and, using (15) and taking logarithms to the base 10,

$$0.4343x_1 = \log (2\pi\lambda\mu) - \log \sin y_1 - \log n. \tag{17}$$

The amplitude of successive fluctuations of $E'(t)$ change in the ratio e^{nx_1}/θ. This damping factor, a, may be written

$$a = e^{2\pi \cot y_1}. \tag{18}$$

We may now insert the empirically calculated values of the parameters. The unit in which t is measured is 3 months. Details of the method used in estimating the parameters are given in the appendix at the end of this paper. The values taken are:

$\alpha = 0.575$,	$\lambda = 1.667$,
$\gamma = 0.430$,	$\mu = 3.322$,
$\delta = 0.100$,	$\theta = 7$ or 8.

For the moment we may keep θ variable. Inserting the values of the other parameters in (17) we obtain:

$$0.4343x_1 = 1.5421 - \log \sin y_1 - \log n, \tag{19}$$

and also

$$0.4343x_1 = 0.4343y_1 \cot y_1. \tag{20}$$

Table 1 shows values of x_1, n, θ, a, which correspond to various values of y_1 between 60° and 120°. The "best" value for θ corresponds to a value for y_1 between 84° and 88°, which makes the period of the cycle between $7\frac{1}{2}$ and $8\frac{1}{4}$ years. Such a solution gives the damping factor, a, a value greater than unity. An undamped cycle of $8\frac{3}{4}$ years would fit our data, provided that a less good value (8.71) was given to θ. While it is unnecessary to assume that there is no damping,[7] it is not unrea-

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R. W. James and M. H. Belz, "The Significance of the Characteristic Solutions of Mixed Difference and Differential Equations," ECONOMETRICA, Vol. 6, October, 1938, pp. 326–343.

[7] Cf. however, Kalecki, "A Macrodynamic Theory of Business Cycles," ECONOMETRICA, Vol. 3, July, 1935, p. 336, where such an assumption is made.

sonable to assume that a should be neither excessively large nor excessively small. Thus, the condition

(21) $$\tfrac{1}{2} < a < 2$$

gives

(22) $$7 < \theta < 11\tfrac{1}{4},$$

and

(23) $$30 < n < 42 \text{ (n measured in quarters)},$$

or

(24) $$7\tfrac{1}{2} < n < 10\tfrac{1}{2} \text{ (n measured in years)}.$$

TABLE 1

y_1 Degrees	y_1 Radians	$x_1 = y_1 \cot y_1$	$0.4343\, x_1$	$1.5421 - \log \sin y_1$	n	a	θ
60	1.0472	0.604	0.2625	1.6046	21.99	37.50	3.67
70	1.2217	.444	.1932	1.5691	23.76	9.84	4.62
80	1.3963	.246	.1069	1.5487	27.66	3.03	6.15
84	1.4661	.154	.0669	1.5445	30.03	1.93	7.01
88	1.5359	.054	.0233	1.5424	33.05	1.25	8.08
90	1.5708	0	0	1.5421	34.84	1.00	8.71
92	1.6057	−0.056	−0.0243	1.5424	36.87	0.80	9.42
96	1.6755	− .176	−.0765	1.5445	41.78	.52	11.14
100	1.7453	− .308	−.1337	1.5487	48.12	.33	13.37
110	1.9199	− .699	−.3035	1,5691	74.57	.10	22.79
120	2.0944	−1.209	−.5250	1.6046	134.8	.03	44.91

Within these limits, any solution chosen gives a value of n which is in accord with the facts, and a value of θ which fits the data. Indeed, since any value of θ much less than 7 or much more than 10 gives a bad fit for (6), the "best" solution is perhaps in the neighborhood of $y_1 = 88°$, giving $a = 1.25$, $\theta = 8$, and $n = 33$ quarters, or $8\frac{1}{4}$ years.

An important problem to consider is the effect on n and a of changes in the various constants, due either to changes in actual economic conditions or to errors of estimation of the constants themselves. We shall first examine the effects of changes in $\lambda\mu$ assuming θ to remain unchanged. We shall also assume $\pi/3 < y_1 < 2\pi/3$ and $\lambda > 0$, $\mu > 0$.

Differentiating (15) and (18) partially with respect to y_1, we have

(25) $$\partial n/\partial y_1 = -\, 2\pi\theta/y_1^2 < 0,$$

(26) $$\partial a/\partial y_1 = -\, 2\pi \operatorname{cosec}^2 y_1 e^{2\pi \cot y_1} < 0.$$

Also (16) and (17) give

(27) $$\log 2\pi + \log (\lambda\mu) = 0.4343 y_1 \cot y_1 + \log n + \log \sin y_1$$

or

(27a) $\log (\lambda\mu) = 0.4343 y_1 \cot y_1 + \log \sin y_1 - \log y_1 + \log \theta.$

$$(28) \quad \therefore \quad \frac{1}{\lambda\mu}\frac{\partial(\lambda\mu)}{\partial y_1} = 1.4343 \cot y_1 - \frac{1}{y_1} - 0.4343 y_1 \operatorname{cosec}^2 y_1.$$

The expression to the right of (28) is negative within the range. Hence $\partial n/\partial(\lambda\mu)$ and $\partial a/\partial(\lambda\mu)$ are positive.

In the appendix it is shown that it is unlikely that $\lambda\mu$ lies outside the range $0<\lambda\mu<7.5$. Values of a, n corresponding to values of $\lambda\mu$ within the range and to various values of θ are shown in Table 2.

TABLE 2

	y_1	n	a	θ
$\lambda\mu=1$	110° 120° 125°	13.45 24.31 35.34	0.10 .03 .01	4.11 8.10 12.27
$\lambda\mu=3.5$	90° 100° 110°	21.57 29.79 46.16	1.00 .33 .10	5.39 8.27 14.10
$\lambda\mu=5.5$	80° 90° 100°	27.66 34.84 48.12	3.03 1.00 .33	6.15 8.71 13.37
$\lambda\mu=7.5$	70° 80° 90°	32.13 37.41 47.12	9.84 3.03 1.00	6.25 8.31 11.78

It is evident that values of $\lambda\mu$ in the neighbourhood of 5.5 (the value actually chosen) enable θ to be varied considerably without obtaining improbable values for a. The values of n corresponding to values of θ between 6 and 9 are acceptable for values of $\lambda\mu>3\frac{1}{2}$. Smaller values of $\lambda\mu$ require values of θ greater than, say, $7\frac{1}{2}$, if satisfactory values for n are to be obtained.

We may finally examine the effects of changes in the original constants, α, γ, δ, λ. We assume that $0<\delta<1$, $0<\gamma<\alpha<1$ and $\lambda>0$. Evidently, from (10a), $\partial(\lambda\mu)/\partial\gamma$ and $\partial(\lambda\mu)/\partial\delta$ are negative, and $\partial(\lambda\mu)/\partial\alpha$ is positive. Also $\partial(\lambda\mu)/\partial\lambda$ is positive, since the assumptions make $\mu>0$.

Of the constants, the one most liable to serious error is α. The following table shows values of $\lambda\mu$ corresponding to various values of α assuming the values of the other constants to be those already given. The estimated value of α is 0.575 ± 0.175.

$\alpha=0.4$	0.5	0.575	0.65	0.7	0.75
$\lambda\mu=0.54$	3.77	5.55	6.74	7.45	8.07

We may deduce that values of α between 0.5 and 0.7 lead to reasonable values of n.

The effects of errors in δ, the next most doubtful of our constants, can be similarly shown (assuming the other constants to keep their assigned values). The estimated value of δ is 0.100 ± 0.020.

$\delta = 0.080$	0.100	0.120
$\lambda\mu = 6.50$	5.55	4.75

Variations in λ between 1.3 and 2 lead to variations in $\lambda\mu$ between 4.5 and 6.5. The estimate of γ appears to be so close (0.430 ± 0.006), that no appreciable error in $\lambda\mu$ is caused by errors in the estimation.

If we refer briefly to α, λ as "investment constants" and γ, δ as "savings constants" we may summarise our conclusions as follows:

(a) Increases in the investment constants increase the period of oscillation and the damping factor.

(b) Increases in the savings constants decrease the period of oscillation and the damping factor.

(c) An increase in the investment lag (θ) increases the period of oscillation and decreases the damping factor.

The conditions that there will be no oscillations are:

(a) $\alpha < \gamma(1-\delta)$: in this case $\lambda\mu/\theta < 0$;

(b) θ is small enough, and/or the investment constants are large enough, and/or the savings constants are small enough: the exact condition is that $\lambda\mu/\theta \geqq e$.

A slightly modified solution can be obtained if it is assumed that the investment lag (θ) is different in the upswing and the downswing of the cycle, the changes in its value taking place, let us say, suddenly at or near the turning points. An examination of the actual data shows that a larger value of θ fits the data better for 1930–1933, and a smaller value of θ for 1934–1936.

If this is so, we should have the tendency of a long cycle with a small damping factor in the downswing, and a short cycle with a large damping factor in the upswing. Thus, keeping $\lambda\mu = 5.55$, and assuming $\theta = 10$ for the downswing and $\theta = 7$ for the upswing, we have (where n is the period for the *whole* cycle):

For the downswing,	$n = 38$	$a = 0.70$
For the upswing,	$n = 30$	$a = 1.95$

This would lead to a movement with sharp peaks and broad troughs, in which, conceivably, the damping tendency of the downswing might be offset by the anti-damping tendency of the upswing. Such an unsymmetrical movement seems to be not unlike the actual course of many trade cycles.

Another type of movement would be brought about if (assuming θ

constant) the savings constants increased in the downswing, and the investment constants increased in the upswing. In this case, the downswing would again be associated with a smaller damping factor, but, unlike the type of movement just discussed, with a smaller period of oscillation as well. The upswing would lead to larger values for both the period of oscillation and the damping factor.

Such unsymmetrical movements cannot, however, be deduced from our data with any degree of certainty. The facts regarding short-term movements of income, savings, and investment are far too uncertain to provide anything but the broadest generalizations. The results of this paper are based, it must be admitted, on rather insecure statistical foundations. Nevertheless, their agreement with the general features of the recent cycle, which did, evidently, last about eight years, suggests that the original hypotheses are perhaps useful ones.

APPENDIX

Sources of data

These series for E, E_1, E_3, J are estimates of Mr. Colin Clark.[8] E and J are calculated quarterly for 1929–1936; E_1 and E_3 yearly for 1924–1935. The figures for undistributed profits or business savings (E_2) are very close to those of Clark.[9] The figures for individuals' savings, S, are calculated from official figures of savings performed through savings banks, building societies, insurance companies, friendly societies, and national savings certificates. It is assumed that the net savings of individuals taking the form of purchases of new securities were negligible in the period under consideration.

General method of estimating the constants α, δ, γ, λ

The method used was that of correlating the appropriate time series. Regression lines were fitted by the method of least squares on the assumption that the variables J, E_2, and S were dependent. The details and justification of the methods are described in a thesis recently presented by the writer to Oxford University. The calculation of λ was more complicated than that of the other parameters, since it involved the building up of a series for $d/dt[E(t)]$. This series was made as follows:

(a) a three-quarters' moving average was calculated from the quarterly series $E(t)$;

(b) a difference series was calculated from this last series, the terms of which were assumed to be estimates of quantities proportional to $d/dt[E(t)]$.

[8] *National Income and Outlay*, and *Economic Journal*, June, 1937.

[9] *National Income and Outlay*, p. 187. Small adjustments are made to include the effects of companies' debenture interest payments.

The coefficient δ

The estimate of dS/dE_1 (δ) was 0.100 ± 0.020. The range of estimate taken was the standard deviation. Graphical methods indicated a fairly good fit for $\delta = 0.100$, but there was some evidence that a better fit could have been obtained by an equation of the form $\log S = \beta \log E_1 + \text{constant}$. The years taken were 1924–1935.

The coefficient γ

The estimate of (dE_2/dE_3) (γ) was 0.430 ± 0.006. Here the fit was very close indeed. A scatter diagram covering the years 1924–1936 gives practically collinear observations.

The coefficient α

An attempt was made to fit a linear equation connecting J, E_3, and the long-term rate of interest (r) for the period 1927–1935 by the method of multiple correlation. The fit was fairly good, but the coefficient of r in the equation was so small as to make the inclusion of this extra variable superfluous. α was taken as 0.575 ± 0.175. In view of the small number of observations and the comparatively wide range of standard error, the estimate of α is less acceptable than those of γ and δ. Nevertheless the estimate is significant, judged by all the normal tests.

The coefficients λ and θ

Correlations were calculated between the series $I(t)$ and $E'(t-\theta)$, for values $\theta = 2, 3, \cdots, 10$ quarters. The highest correlations (0.80 and 0.83) corresponded to $\theta = 7$ and $\theta = 8$. For $\theta = 4$ and $\theta = 10$ the correlations were 0.60 and 0.61. The ratio of the standard deviations of $I(t)$ and $E'(t)$ was 2:1 with quarterly time units. For $\theta = 8$ the number of quarterly observations was 21, and the value of λ was taken as 1.67. In view of the doubt as to the distribution of errors in the series $E'(t)$, an upper limit of 2 was placed on the value of λ.

The range of $\lambda\mu$

The central value of $\lambda\mu = \lambda/\delta - \lambda\gamma/\alpha\delta + \lambda\gamma/\alpha$ is estimated as 5.55. Its lower limit is probably near zero, since α may be little greater than $\gamma(1-\delta)$. The problem of estimating an upper limit presents some difficulty. If the coefficients have been all underestimated, i.e., if we take the upper limits in each case, the value of $\lambda\mu$ is approximately 7.5. This is probably a reasonable maximum, but upper limits for α, λ and lower limits for γ, δ give a value of 12.

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PERIODOGRAM ANALYSIS WITH THE PHASE A CHANCE VARIABLE*

By EDWARD L. DODD

I. INTRODUCTION

FOR SOME two hundred years, mathematicians, astronomers, and physicists have been interested in periodicities. Among early writers are the famous mathematicians, Euler, Lagrange, Gauss, Bessel, Cauchy, and Fourier. Among later writers should be mentioned in particular Arthur Schuster, who proposed the use of what he called a periodogram for investigating hidden periods. Schuster's methods and ideas form to a large extent the basis of the periodogram analysis of today.[1]

A section of the German encyclopedia written by H. Burkhardt[2] in 1904, with nearly five hundred citations, will give a reader some idea of the extensiveness of the literature on periodicity up to that date. A third of a century has brought forth many, many more papers. It would be difficult to summarize all the proposals that have been made for investigating periods.

For something over a decade papers have been appearing dealing with what in popular language would be called variable periods. Mathematically speaking, a period of time is an interval of fixed length. However, it has seemed reasonable to modify somewhat the conception of a rigidly constant interval of time. Recently, Eugen Slutzky[3] has published in English a summary of this work on random curves. His original work appeared in Russian in 1927. There is also an important paper by Ragnar Frisch,[4] who obtained results of great generality. Dr. Chas F. Roos[5] noted that Frisch's method would require the use of a considerable number of degrees of freedom.

* Paper presented at Colorado Springs, July 27, 1938 before Cowles Commission Fourth Annual Research Conference on Economics and Statistics.

[1] "On Lunar and Solar Periodicities of Earthquakes," *Proceedings of Royal Society*, Vol. 61, 1897, pp. 455–465.

"On the Investigation of Hidden Periodicities with Application to a Supposed 26-day Period of Meteorological Phenomena," *Terrestrial Magnetism*, Vol. 3, 1898, pp. 13–41.

[2] *Encyklopädie der Mathematischen Wissenschaften*, IIA9a, "Trigonometrische Interpolation," pp. 642–693.

[3] "The Summation of Random Causes as the Source of Cyclic Processes," ECONOMETRICA, Vol. 5, 1937, pp. 105–146.

[4] "Changing Harmonics and Other General Types of Components in Empirical Series," *Skandinavisk Aktuarietidskrift*, Vol. 14, 1928, pp. 220–236.

[5] "Annual Survey of Statistical Techniques," ECONOMETRICA, Vol. 4, 1936, pp. 368–381.

If, in line with some recent papers, we wish to modify the notion of an absolutely rigid period, it may be well at first to consider very slight modifications. Such modifications may not describe certain data as accurately as could be done by diverging further from the well-beaten paths. But, in many cases, little or no modification may be needed. By well-known methods a forty-week period in stock prices was established by work done under the auspices of the Cowles Commission.[6] And Benjamin Greenstein[7] found a 9.4-year period in business failures.

The modification which I have in mind may be illustrated as follows: Let us imagine a geyser with a period or quasi-period of 60 min. ± 5 min. By writing ± 5 min., for a probable error, it is implied that, about half the time, the interval between spurts is greater than 55 minutes and less than 65 minutes. Such performance would naturally be regarded as substantially periodic. If, for this postulated geyser, the "errors" are independent, any attempt to forecast the times for spurts two days ahead would be just about futile. The probable error of 5 min. for a single performance would result in a probable error of about a half-hour for a set of 48 performances. For certain phenomena we would, indeed, expect a slipping of phase. But concerning the 11-year sunspot period, Schuster[8] noted that the intensity subsides considerably at times, but the reappearance in *correct phase* substantiates the existence of the period. It is just at this point that the quasi-period that I am considering here differs from a rigid period. There is no permanence of phase. If one performance starts 5 minutes later than expected, there is no more tendency to make up that 5 minutes than to lose another 5 minutes.

It is perhaps obvious that if the permitted slipping of phase is small enough, a theory of quasi-periods on this basis can be framed. We would naturally attempt to secure some generality; but in a first excursion away from strict periodicity, simplicity would be a natural goal. To secure simplicity, it will be assumed that elementary chance deviations, whose cumulative effect determines the phase, are subject to some *symmetric* law of distribution. It will *not* be assumed that these deviations follow the *normal law*, as such an assumption contributes little or nothing to simplicity. And under symmetric functions we can

[6] H. T. Davis, "Significance Tests for Periodogram Analysis with Application to Prices to Common Stocks,"1936, *Abstracts of Papers Presented at the Research Conference on Economics and Statistics Held by the Cowles Commission . . . 1936*, Colorado College Publication, General Series No. 208, pp. 102–103.

[7] "Periodogram Analysis with Special Application to Business Failures in the United States, 1867–1932," ECONOMETRICA, Vol. 3, 1935, pp. 170–198.

[8] "On Sun Spot Periodicities," *Proceedings Royal Society*, A, Vol. 77, 1906, pp. 141–145.

include shock functions, in general inoperative or quiescent but acting sporadically with violence.

II. TESTS WITH DIFFERENCE FUNCTIONS

Let

(1) $$z_1, z_2, \cdots, z_r, \cdots$$

be independent chance variables, each subject to the same symmetric law of distribution, each with mean value zero and variance σ^2.

Let

(2) $$Z_r = z_1 + z_2 + \cdots + z_r.$$

Let k be a quasi-period, θ be $2\pi/k$ radians; α be a constant phase constituent; a be amplitude. Then we take y_r as a quasi-periodic function, where

(3) $$y_r = a \cos (r\theta + \alpha + Z_r), \qquad \theta = 2\pi/k, \qquad r = 1, 2, 3, \cdots.$$

We think of data as made up additively of functions of this type together with functions for trend or additional accidental constituents.

A question which naturally arises is this: If a test function designed to ferret out the exact period k is applied to y_r in (3) with its quasi-period of k, what is the effect? Let us start with a very simple test function

(4) $$Y_r = (y_{r+k} - y_r)^2.$$

It is obvious that if in (3) we set $Z_r=0$, giving to y_r the *exact* period k, then $Y_r=0$; since $k\theta=2\pi$. But the Y_r in (4) will *not* in general be zero when applied to the *quasi*-periodic y_r of (3).

Preparatory to studying Y_r, let us note a few consequences of the assumptions made for z_r and Z_r in (1) and (2). Since the distribution is assumed symmetrical—that is, given by an even function—it follows that the expected value of any odd function such as $\sin z_r$ is zero. Let

(5) $$E(\cos z_r) = \text{Expected } (\cos z_r) = \eta.$$

Then

(6) $$E(\cos Z_r) = \eta^r;$$

(7) $$E[\cos (Z_{r+t} - Z_r)] = \eta^t,$$

where, in the latter, $Z_{r+t}-Z_r$ may be replaced by Z_r-Z_{r+t}. Indeed,

$$\cos Z_r = \cos z_1 \cdot \cos (z_2 + \cdots + z_r) - \sin z_1 \cdot \sin (z_2 + \cdots + z_r).$$

The expected value of the last term is zero; since $E(\sin z_1)=0$, and the

z's are independent. By repeated applications, we are led to η^r. Likewise, treat

$$Z_{r+t} - Z_r = z_{r+1} + \cdots + z_{r+t}.$$

If, now in (3), we set

$$\beta_r = r\theta + \alpha + Z_r, \qquad Z_k' = Z_{r+k} - Z_r; \tag{8}$$

we may, since $k\theta = 2\pi$, write Y_r in (4) as follows:

$$\begin{aligned} Y_r &= [a \cos (\beta_r + Z_k') - a \cos \beta_r]^2 \\ &= a^2 + (a^2/2) \cos (2\beta_r + 2Z_k') + (a^2/2) \cos (2\beta_r) \\ &\quad - a^2 \cos Z_k' - a^2 \cos (2\beta_r + Z_k'). \end{aligned} \tag{9}$$

But, by (7),

$$E(\cos Z_k') = \eta^k.$$

Hence

$$E(Y_r) = a^2(1 - \eta^k) + R_r, \tag{10}$$

where R_r is the sum of the terms involving β_r. The average value of R_r for k consecutive values of r would presumably be close to zero; likewise for any fairly large number of values of r with the r's running consecutively in broken sets.

If R_r is ignored, the expected value of Y_r becomes $a^2(1-\eta^k)$. Now if the distribution for z_r has a very small standard deviation, then there is high probability that z_r will be close to zero, and thus that $\cos z_r$ will be close to unity; and thus from (6), that η will be but a trifle less than unity. In this case, $E(Y_r) = a^2(1-\eta^k)$ almost vanishes. In the lines that follow (4), it was noted that $Y_r = 0$ if y_r has rigidly the period k. On the other hand, if η^k is small—either because k is very large or because η itself is small, then $E(Y_r) = a^2$, nearly. This indicates that the test function Y_r in (4) has now lost nearly all its potency. For the average value of $(y_r - y_{r+t})^2$ is also a^2, where t need not be k, but is *any integer* taken at *random.*

For testing periods, the function Y_r in (4) is one of the simplest imaginable. This does not imply, of course, that Y_r is the most effective or that it is the simplest. Indeed, from the standpoint of computation, a still more simple function was used in a recent paper by Dinsmore Alter.[9] To test data, $x_1, x_2, x_3, \cdots$, for a period of l, Alter finds the average value of $|x_i - x_{i-l}|$. He thus uses absolute values, instead of squares of differences. By this method, he considers that he has ob-

[9] "A Simple Form of Periodogram," *The Annals of Mathematical Statistics*, Vol. 8, 1937, pp. 121–126.

tained "perhaps the most definite single piece of evidence ever found for rainfall cycles. Outstanding is a cycle of about 45 years with its fourth harmonic as the secondary feature," for the north California coast. On the theoretic side, absolute values of differences are not as simple as squares of differences.

If k is an even integer, then another test function very similar to (4) may be set up. We now take

$$Y_r' = (y_{r+k/2} - y_r)^2. \tag{11}$$

The effect of this Y_r' is, however, opposite to that of Y_r in (4). For Y_r applied to a sequence with period k is a smoothing operator, reducing variance and thus standard deviation; whereas Y_r' tends to increase variance. As applied to the quasi-periodic function y_r in (3), the final result bears a resemblance to (10). It is

$$E(Y_r') = a^2(1 + \eta^{k/2}) + R_r', \tag{12}$$

where R_r' would under general conditions have average value about zero. As the span is now $k/2$ instead of k, it is natural that $k/2$ should be the index of the power of η. Moreover, $\cos(\alpha' + k\theta/2) = -\cos\alpha'$, since $k\theta = 2\pi$; and this accounts for the positive sign in the parentheses in (12). Here, if the phase disturbance z_r has small variance, $\eta = 1$, nearly; and approximately $E(Y_r') = 2a^2$; and Y_r' is then a fairly strong operator. With $\eta^{k/2}$ small, Y_r' becomes impotent.

For the special case of $k = 4$, we need only put together two operators a quarter-wave apart, to obtain a harmonic operator, in line with the Schuster method of testing for periods.

III. HARMONIC TEST FUNCTIONS

We come now to harmonic operators. As a preliminary step in testing for an integral period of k, the data are often set down in rows, with k items in a row. In a single set, suppose that there are K items—where K is a multiple of k. Then, let

$$\theta = 2\pi/k, \qquad t = mK, \qquad m = 0, 1, 2, \cdots; \tag{13}$$

$$c_r = \sqrt{2/K}\cos r\theta, \qquad s_r = \sqrt{2/K}\sin r\theta; \tag{14}$$

$$C_m = \sum_{r=t+1}^{t+K} c_r y_r, \qquad S_m = \sum_{r=t+1}^{t+K} s_r y_r, \qquad I_m = C_m^2 + S_m^2. \tag{15}$$

To obtain, first, the expected value of I_m, write

$$\begin{aligned} I_m &= (c_{t+1}y_{t+1} + \cdots + c_{t+k}y_{t+k})^2 + (s_{t+1}y_{t+1} + \cdots + s_{t+k}y_{t+k})^2 \\ &= (2/K)(y_{t+1}^2 + \cdots + y_{t+k}^2) + (4/K)\sum y_i y_j \cos(i - j)\theta, \end{aligned} \tag{16}$$

where i and j take values from $t+1$ to $t+K$, subject to $i>j$. But, with y_r given by (3),

$$2y_iy_j = a^2 \cos\left[(i+j)\theta + 2\alpha + Z_i + Z_j\right] + a^2 \cos\left[(i-j)\theta + Z_i - Z_j\right].$$

For the last term, we have, by (7),

$$E\left\{a^2 \cos\left[(i-j)\theta + Z_i - Z_j\right]\right\} = a^2\eta^{i-j} \cos (i-j)\theta. \tag{17}$$

Noting that $y_i^2 = a^2/2+(a^2/2) \cos (2i\theta+2\alpha+2Z_i)$, we may from (16) and (17) write

$$E(I_m) = a^2 + (2a^2/K)\sum_{t=1}^{K-1} (K-s)\eta^s \cos^2 s\theta + R, \tag{18}$$

where R is a "remainder" consisting of terms which may be positive or negative, and which under some general conditions would have an average value of about zero. The above value of $E(I_m)$ with the omission of R may be called the principal part of $E(I_m)$. When η approaches unity, this principal part approaches $Ka^2/2$ as a limit, the known value of I_m when a chance phase is not being considered.

For C_m and S_m in (15) we may obtain expected values, reckoned at the time when r takes on the value $t+1$, as follows:
Set

$$\alpha' = \alpha + Z_{t+1}, \qquad H = 1 + \eta + \eta^2 + \cdots + \eta^{K-1}. \tag{19}$$

Then

$$\begin{aligned} E(C_m) &= aH\sqrt{1/2K} \cos \alpha' + R_1, \\ E(S_m) &= aH\sqrt{1/2K} \sin \alpha' + R_2, \end{aligned} \tag{20}$$

where R_1 and R_2 are remainders.

To illustrate (18), suppose that z_r is normal with variance σ^2. Then

$$\eta = E(\cos z_r) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} \cos z e^{-z^2/2\sigma^2}dz = e^{-\sigma^2/2}. \tag{21}$$

Suppose now that $\sigma=0.2$ radian $=11.°46$, making the probable error about 8°. Then $\eta=0.9802$. If now $k=8$, and $K=2k=16$, then, from (18), $E(I_m)=7.23a^2$, if we ignore R. This is about 90 per cent of the value of I_m, given by $Ka^2/2$, when there is no phase irregularity assumed. If, however, we take 4 cycles, and set $K=4k=32$, then $E(I_m)=13.09a^2$, only about 82 per cent of the value $32a^2/2$. If, as here taken, the chance element z_r has a probable error of about 8°, it would not indeed be unusual for 4 cycles of the 8-year "period" to take 33

years. From (10), (12), (18), (20), and similar formulas for expected values, we can estimate to what extent various test functions will be damaged by a chance phase element. It is obvious that the larger the variance σ^2, the smaller must be the set of K consecutive terms that we can profitably group together in testing for periods.

IV. CONCLUSION

The foregoing is but a start in the general problem of the effect of an assumed chance phase upon the results of period testing. The data are here considered as the additive result of quasi-periodic functions together with other constituents such as trend elements or additional chance elements. If these chance elements are considered as independent of the chance elements affecting phase directly and resulting in quasi-periodicity, then there seems to be no great difficulty in separating out the effects of the additional chance elements, when certain test functions are used. A good many important questions, however, have not been taken up. But, under certain assumptions, the formulas for expected values as obtained above seem to form a basis for estimating the decrease in effectiveness or efficiency of certain test formulas if in the phase there exists a chance element, cumulative in the sense postulated in (2), and subject to some symmetric law of distribution, not necessarily the normal distribution.

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FULL EMPLOYMENT WITH A NONHOMOGENEOUS LABOUR FORCE

By Henry Smith

I

The following pages embody an attempt to relate the general corpus of reasoning inspired by Mr. Keynes' *General Theory* to the conditions of a labour market in which labour is specialised, relatively immobile between industries, and unsuitably distributed. It will proceed throughout upon one minor and two major premises. The major are:

(1) That the stimulation of secondary employment in the consumers'-goods industries resulting from an increase in employment in the capital-goods industries does not depend directly upon changes in real wages, and is compatible with unchanged real earnings. Lessened real wages are only operative through their effect upon the marginal efficiency of capital, and an increase in investment may accompany a stable or falling marginal efficiency of capital, given the appropriate action upon the part of the monetary authorities.

(2) That changes in the cost of living to the working classes will result in changes in money wages, which, though neither simultaneous nor of an exactly equal order of magnitude, will tend to precede in time and to exceed in magnitude the tendencies towards increased employment resulting from rising prices in the consumers'-goods industries. This implies that any attempt to induce secondary employment in the capital-goods industries will be directly inflationary. It follows that the doctrine of the multiplier is only tenable when considered as the relationship between primary employment in the capital-goods industries and secondary employment in the consumers'-goods industries.

The minor premise is that the monetary authorities take whatever steps are necessary to render possible any increase in investment which the argument postulates.

II

Our purpose is to investigate the possibility of ascertaining, with some degree of accuracy, the limits to which a policy of eliminating unemployment by means of low interest rates and public works can be carried without reaching the point of inflation. The argument will be confined to a short period; it will not consider the problems arising out of the cumulative fall in the rate of interest which the continuance of a successful policy of this type would in all probability engender.

Figure 1 is instrumental in defining some terms which it will be found serviceable to employ, and in exemplifying the conditions of full em-

ployment with a perfectly homogeneous and mobile labour force. Along the OY axis we measure the average wages cost of production: only wages are considered for the present. Thus the line EI indicates the average wages cost per unit of output as a whole. The line CI' indicates the proportion of the average wage cost of each unit of aggregate output which represents wage costs in the capital-goods industries. As we assume the whole labour supply to be homogeneous, wages per hour in the capital-goods and consumers'-goods industries are identical.

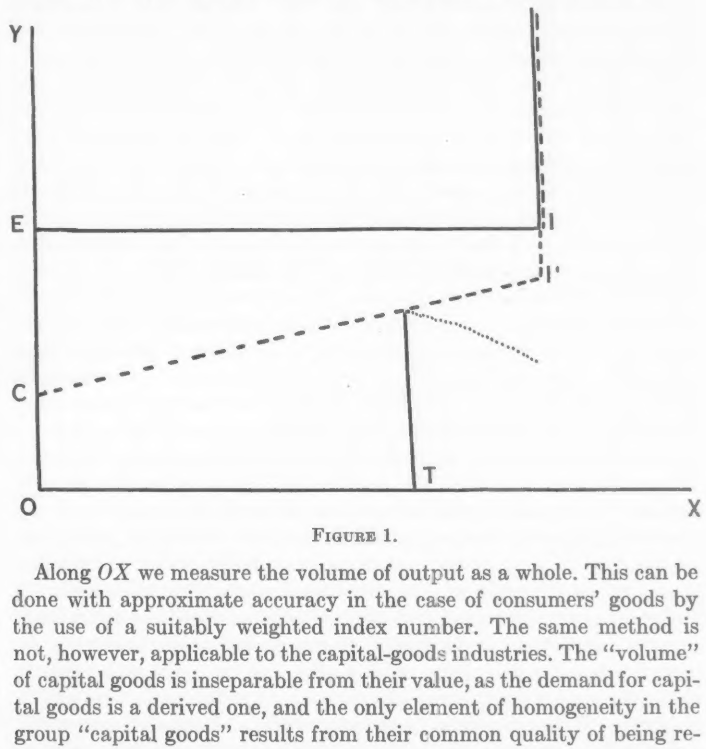

FIGURE 1.

Along OX we measure the volume of output as a whole. This can be done with approximate accuracy in the case of consumers' goods by the use of a suitably weighted index number. The same method is not, however, applicable to the capital-goods industries. The "volume" of capital goods is inseparable from their value, as the demand for capital goods is a derived one, and the only element of homogeneity in the group "capital goods" results from their common quality of being regarded as revenue-earning assets. We must therefore measure the "volume" of the output of capital goods by representing it as part of the volume of consumers' goods, allowing for it at whatever price ratio may obtain between the two classes. Thus we measure the volume of output as a whole along OX in units of an index number of consumers' goods, a varying proportion of which represents the output of capital goods.

We can now define the conditions of full employment, and trace the steps by which it is approached. As the volume of employment and of output increases, the propensity to consume diminishes, and the multiplier (OE/OC) decreases. The marginal efficiency of capital declines, and consequently, in order to allow the expansion to continue, the rate of interest is reduced, to the accompaniment of the appropriate changes in the quantity of money.

At this point it becomes necessary to consider the effect of this decline in the rate of interest upon the volume of the output of capital goods as calculated. It will, of course, enhance the value of all pre-existing capital goods, but it will not affect the price level of contemporary output if the conditions of supply are perfectly elastic, as is here supposed. The efficiency of capital is the sum of the efficiencies of specific capital assets. If the rate of interest is reduced, demand for the goods next upon the combined investment preference scale of entrepreneurs will be stimulated, at existing prices. If there is nothing in their conditions of supply to cause the prices of the specific goods concerned to rise, then the fall in the rate of interest will not affect prices. (Should the price of capital goods rise, their marginal efficiency at existing prices of final products therefore falling, a suitable reduction in the rate of interest can enable demand for them to be maintained. But the only way in which changes in the rate of interest can affect the price level of the contemporary output of the capital-goods industries is by placing a greater or lesser amount of money at the disposal of entrepreneurs, and by making them more or less willing to employ it.)

Consequently, as we move from left to right across the diagram, the multiplier diminishes, the marginal efficiency of capital and the rate of interest decline together, while the level of money wages per hour and per unit of output remain unaltered. There are, moreover, no changes in the relative prices of consumers' goods and capital goods, nor in the price level of output as a whole. But when we reach the point of total employment, the multiplier suddenly becomes zero; i.e., an addition to employment in the capital-goods industries no longer stimulates secondary employment, but can only take place at the expense of a contraction of employment in the consumers'-goods industries. We have arrived at the verge of Mrs. Robinson's precipice.

Let us take note of some of the features of this interesting and dangerous situation. We have reached the point of *total* employment—all our labour force is now employed. The multiplier has become equal to zero. Now there is no doubt that we have also reached the point of "full employment," and that we are also on the eve of an "inflation." But both "full employment"—which is reached on the verge of "inflation"—and inflation itself, are difficult to define under less rigorous

assumptions. A situation in which something vague is defined as the immediate preliminary to something indefinite, but disastrous, is somewhat unsatisfactory. We know that full employment must be reached in an imperfect labour market before total employment, we know that any attempt to expand the output of a fully employed community by manipulating the rate of interest must lead to inflation, but we can only define the one term in terms of the other.

In our working model of a perfect labour market, however, we found that indubitable inflation and perfectly full employment coincided with the fall of the multiplier to zero. Let us therefore examine the possibilities of the multiplier itself as a definant of full employment and of inflation.

III

Figure 1 may also be employed to depict the conditions of a labour market in which there is perfect mobility *within* the capital-goods industries as a class, and within the consumers'-goods industries class, but in which there is no transfer of employment between the two classes. In no other way are the assumptions of the previous section relaxed.

As output and employment expand, a point will now be reached (unless labour is perfectly distributed between the two specialised classes) at which a shortage of workers becomes apparent in the one while unemployment continues in the other, and wages will therefore tend to rise in the former class. Let us assume that this point, T, is first reached in the capital-goods industries; if investment is to continue the rate of interest must fall sufficiently to offset the diminution in the marginal efficiency of capital caused by the rising prices of specific capital assets. In so far as the rise in prices is solely due to the shortage of suitable labour, prices will only rise in the same proportions as wages. Thus wages cost per unit of aggregate output will remain stable: the rising prices of capital goods will cause each unit to count as an increasing multiple of the equivalent consumers' goods in which it is measured.

But a change may take place in the multiplier. If the multiplier be defined as the ratio in which *employment* in the capital-goods industries increases *employment* as a whole, it will cease to shrink and will follow the course of the dotted line. Each man employed in the capital-goods industries will have more to spend on consumers' goods, and although the propensity to consume will be lessened for persons deriving income from the capital-goods industries as a class, it is probable that the net result of increased earnings and decreased propensity to consume will be increased secondary employment. But our method of presentation

prevents this from becoming apparent on the diagram, because we represent the *proportions* of each type of earnings per unit of aggregate output; as the earnings per physical unit of capital goods rise the rising value of the output causes its "volume" to increase. Consequently there is little reason to expect the multiplier as depicted to alter its trend.

What does this imply? It appears that we have imperceptibly passed over to a redefinition of the multiplier as the ratio of *earnings* in the capital-goods industries to earnings as a whole. As this new definition is compatible with the argument and conclusions of the previous section, we shall henceforth adhere to this more general form. But one important change in our terminology now becomes essential. Arguing in terms derived from the concept of the relation between primary and secondary employment measured in numbers of men, we defined full employment as the position in which the multiplier is equal to zero; i.e., in which any increase in employment in the capital-goods industries could only lead to a decrease of similar magnitude in the consumers'-goods industries. The new definition, however, relating expenditure upon wages in the capital-goods industries to expenditure upon wages in the consumers'-goods industries, gives us an infinite multiplier when full employment is reached. Any attempt to further increase investment, backed by a compatible monetary policy, would only serve to drive up the prices of consumers' goods, hence wages, hence prices. . . . Thus the rise in expenditure upon wages in the consumers'-goods industries would be cumulative, and would become infinite were the rate of investment to be maintained.

It has also become necessary to divorce the concept of total employment from that of full employment, and both of them from that of any given output measured in stable terms between capital and consumers' goods. Full employment is now only definable by reference to the multiplier: it is the point at which the multiplier becomes zero in terms of employment and infinity in terms of expenditure.

IV

Let us now reverse the special conditions of the previous section, and assume that, while labour is still divided into two internally homogeneous groups between which transference is impossible, it is in the consumers'-goods group in which the relative scarcity occurs. As employment increases, with a declining multiplier and a falling rate of interest, a point will be reached at which the supply of labour in the consumers'-goods group will be exhausted, while workers in the capital-goods industries are still seeking employment. What will be the results of any increase in investment beyond this point? Mr. Hicks

argues that prices of consumers' goods will rise, all wages will rise in consequence, and general inflation will develop.[1] Mrs. Robinson, on the other hand, considers that increased profits in the consumers'-goods industries, during the period in which wages lag behind prices, and the desire to substitute capital for labour after wages have caught up, will combine to increase the marginal efficiency of capital.[2] Thus each increase in investment will cause secondary employment in the only field where it is now possible, the capital-goods industries.

It must be remembered, however, that neither of these conclusions was reached upon the basis of the somewhat rigorous assumption we have been employing: it is not without interest to compare them with the results obtained from our "working model." If labour within the consumers'-goods industries is perfectly homogeneous, and the point of full employment has been reached, no further expenditure upon wages can increase output. Thus any increase in investment which increases the demand for consumers' goods can only drive up their price. Wages will rise: in the consumers'-goods industries, fairly soon; in the capital-goods industries, where unemployment still persists, more slowly or not at all. Each resulting investment commenced (substitution of capital for the now scarce types of labour) would, by the same mechanism, cause a further increase in wages in the consumers'-goods industries. Moreover, real wages in the capital-goods industries are not infinitely reducible; if prices continue to rise a point will be reached at which either money wages will be increased or the supply of labour will be curtailed. Each increase in wages in the capital-goods industries will still further increase the prices of consumers' goods, and so accelerate the whole process.

Thus the multiplier becomes infinity, and the point of full employment is reached, at the point of total employment in the consumers'-goods industries; any further increase in the expenditure upon investment will commence an infinite increase in the wages bill of the consumption-goods industries, although labour is still unemployed in the capital-goods industries.

The only conditions upon which secondary employment in the capital-goods industries (in Mrs. Robinson's sense) is possible within the limits of our assumptions are either that wages in the consumers'-goods industries should not rise, or, if they do, they should rise so slowly that the capital assets which the entrepreneurs' profits have induced them to purchase shall be in operation, increasing output, before they rise. Otherwise each commenced investment will still further cause prices and wages to rise and the process will be cumulative.

[1] "Mr. Keynes' Theory of Employment," *Economic Journal*, June, 1936.
[2] *Essays in the Theory of Employment*, pp. 40–60.

Let us now, armed with these conclusions, reapproach the multiplier, which we have redefined as the proportional relationship between the wages cost of a small increase in investment and the total increase in the wages bill resulting therefrom. The multiplier depends in part upon the propensity to consume: provided that any part of an addition to income is consumed, its magnitude does not affect the argument. Thus when a small sum, Y, is spent upon capital goods, and the propensity to consume is $I-X$, the multiplier will be $[Y(I-X)(1/X)]/Y$, i.e., will be wholly dependent upon the propensity to consume, while (writing E for the elasticity of supply of labour) $E=\infty$ in both the capital- and the consumers'-goods industries. And the same will be true if $E=\infty$ in the consumers'-goods industries meanwhile becoming $<\infty$ in the capital-goods industries. If $E<\infty$ in the consumers'-goods industries, however, the propensity to consume does not wholly determine the multiplier: an increase in investment will then cause a cumulative increase in the wage bill of the consumers'-goods industries.

We have already defined the point of full employment, in a manner applicable to all three of the situations which we have discussed, as the point at which the multiplier becomes infinite: this can now be defined in the following manner. The multiplier becomes infinite when the propensity to consume is any positive fraction and when, writing E for the elasticity of supply of labour in the capital-goods industries and e for the consumers'-goods industries, $e/E=0$. In our first example E and e simultaneously became $=0$. In our second, the rising prices of capital goods, offset by decreases in the rate of interest, allow production to continue smoothly over the range of output where $e/E=\infty/0$, and the final position is identical with that of example one. In our third, where $e=0$ while $E=\infty$, $0/\infty=0$.

We can now, therefore, abandon the idea of the multiplier and concentrate upon the relative elasticities of supply of labour in the two groups of industries. In our simple model the nearer e/E approached to 0 the nearer "inflation" approached. And we have defined both full employment and inflation in terms of a third concept, that of the ratio of the elasticities of supply of two classes of labour. It now remains to enquire if the argument of our simple model can be applied to the real world.[3]

[3] The preceding analysis lays no claim to originality. The distinction between the income multiplier and the employment multiplier is of course present, in a much more subtle and sophisticated form, in the *General Theory* itself; most of the other concepts employed occur, directly or by implication, in the literature of the subject. Those aspects of the subject which seem to lend themselves to mensuration have been abstracted and, without overburdening the text with reference and acknowledgement, fitted into a suitable framework.

V

In Great Britain in the twentieth century the labour forces of the consumers'-goods and the capital-goods industries are neither distinct from each other nor internally homogeneous. Indeed the line of demarkation between the two classes of industry is hard to draw, as the one shades off imperceptibly into the other. Therefore it is necessary, as a preliminary, to ascertain if the two groups have distinct and different patterns of behavior, before attempting to interpret that behavior in the light of our analysis. For this task we rely upon Mr. Ramsbottom's estimates of full weekly earnings in the month of December, which are available over a number of years.[4] They do not cover all industries, and some of the industries included are not suitable for our purpose, either because comparable employment figures are not obtainable, or because the occupations concerned cannot by any stretch of the imagination be wholly included in either of our two categories. None of the employment figures derivable from the Ministry of Labour *Gazette* are indeed perfectly comparable, but it is possible to produce figures which are sufficiently satisfactory from both points of view to serve our purpose.

The capital-goods-industries sample is composed of the following: Coal Mining, Iron Mining, and Other Mining; Heavy Chemicals, Iron and Steel, General Engineering, Tinplates and Tubes, Shipbuilding, Electrical Cables, Light Castings, and Other Metal Industries; Jute, Building, Public Works, Electrical Installation, Cement, Brickmaking, and Coke Ovens. In 1932 these industries accounted for the employment of approximately 2,635,000 persons.

The consumers'-goods-industries sample is composed of the following: Soap and Candles, Seeds and Oils, Paint Colour and Varnish, and Brass Wares; Cotton, Woollen and Worsted, Hosiery, Carpets, Flax, Lace, Silk, and Textile Finishing; Tailoring, Dressmaking, Hats and Millinery, Boots, and Laundry Services; Flour Milling, Baking, Sugar and Preserving, Brewing, and Tobacco; Paper Manufacturing, Paper Bags and Boxes, Printing and Bookbinding, and Pottery; Tanning, Leather Goods, and Furniture. In 1932 these industries accounted for the employment of approximately 2,451,000 persons.

The first task is to find out how employment in these two groups of industries has varied in recent years. Table 1, which is constructed from the Ministry of Labour returns, gives employment upon a series of comparable dates. To compare with this we need an index of wage rates for each of the two groups. These appear in Table 2 and are constructed by weighting the rates quoted by Mr. Ramsbottom for each

[4] E. C. Ramsbottom, "The Course of Wage Rates in the United Kingdom," *Journal of the Royal Statistical Society*, 1935, No. IV, 1938, No. I.

TABLE 1
EMPLOYMENT IN CAPITAL-GOODS INDUSTRIES (19) AND IN CONSUMERS'-GOODS INDUSTRIES (28)
Unit: 1000 persons

		Capital Goods	Consumers' Goods
1932	June	2,635.0	2,451.0
	October	2,585.6	2,501.1
	December	2,630.2	2,513.2
1933	February	2,596.8	2,460.3
1934	June	2,933.0	2,574.0
	October	2,991.7	2,537.1
	December	3,211.8	2,534.8
1935	February	3,157.8	2,492.3
	June	3,061.0	2,579.0
	October	3,147.2	2,555.5
	December	3,169.0	2,561.4
1936	February	3,123.2	2,536.5
	June	3,281.0	2,649.0
	October	3,356.9	2,630.1
	December	3,348.4	2,626.6
1937	February	3,372.9	2,637.1
	June	3,680.0	2,927.0
	October	3,632.5	2,702.2
	December	3,507.7	2,613.0
1938	February	3,526.3	2,448.3

industry or service by the number employed in 1932. We can now proceed to estimate the elasticity of supply of labour in the two groups on a series of different bases; the results of this process appear in Table 3. Four methods are adopted: in all cases the wage rates employed are for the end of December; with them we compare employment in the previous June, in the previous October, in December, and in the subsequent February.

TABLE 2
WAGE INDICES IN THE CAPITAL-GOODS AND CONSUMERS'-GOODS INDUSTRIES

	1932	1934	1935	1936	1937
Capital Goods	100.0	99.8	101.1	104.3	109.4
Consumers' Goods	100.0	100.0	100.4	102.3	106.7

It is extremely doubtful if the "arc" elasticity over the whole period, in whatever manner it be measured, is of any significance at all, beyond the drastic manner in which it illustrates the downturn of economic activity in the United Kingdom in the winter of 1937 and the spring of 1938. Between June and February the decline in employment in the consumers'-goods industries (which also serve foreign markets, in which the decline in demand was most marked) was sufficient to turn an increase of 19.4 per cent in 1932 into a decrease. But it is in

TABLE 3
ELASTICITY OF SUPPLY OF LABOUR IN THE CAPITAL-GOODS AND CONSUMERS'-GOODS INDUSTRIES

	Capital Goods				Consumers' Goods			
	A	B	C	D	A	B	C	D
1932–37	4.22	4.31	3.55	3.81	2.90	1.20	0.60	−0.07
1932–34	∞	∞	∞	∞	∞	∞	∞	∞
1934–35	3.96	4.91	−1.22	−1.00	0.05	0.02	3.42	4.42
1935–36	2.25	2.08	1.77	1.50	1.42	1.53	1.13	2.08
1936–37	2.53	1.71	0.97	0.94	2.44	0.64	−0.01	−1.66

A = June employment, wages subsequent December.
B = October employment, wages subsequent December.
C = December employment and wages.
D = December wages, and employment in subsequent February.

the year-to-year adjustments that the important evidence and the distinctive patterns of conduct of the two groups becomes apparent. Over the period from 1932 to 1934 the elasticity of supply of both types of labour was infinite; as one would expect in the very earliest stages of recovery from a depression in which unemployment had been universal and in which the decline in prices had been sufficiently precipitous to increase real wages to a considerable extent, it was possible to increase employment over a wide and representative field of industry without raising wages at all, and indeed in some few cases in face of a slight decline in money wages. Between 1934 and 1935 the changes in the capital-goods industries are clearly defined; the proportionate increase in employment during June and October over the previous June and October was considerably greater than the increase of wages in December over those of the previous year, but in both December and February there was an actual decline over the course of the year. In the consumers'-goods industries there was a slight increase in employment in June, followed by an infinitesimal decline in October and a substantial advance in both the following periods. For both the next two annual periods employment in the capital-goods industries follows a similar course; in the June of each year an advance is made over the employment of the previous June, greater in proportion than the December rise in wages, but the decline in this proportional advance for each of the succeeding recorded periods of the same year is betrayed by a steady decline in the elasticity of supply. In the consumers'-goods industries during 1936 there was a steady proportionate increase except for a setback in December, and during the following year an even more steady proportionate decline.

Which of these sets of employment figures are the ones relevant for our purpose? We wish to measure the elasticity of supply of labour, the

relationship between increases in employment and increases in wages. Now there are two reasons, one theoretical and one practical, which combine to indicate that we should choose the relationship between year-to-year changes in employment at dates chosen comparatively early in the year, and our December-to-December wage changes. The state of employment is almost certainly the major determinant of wage rates, but many other things besides wage rates play a part in determining the level of employment. Thus, in order to ascertain the wage rates associated with any change in the level of employment in an industry, we must allow for the lapse of sufficient time for the effect of that change to have worked itself out; very probably by that time the level of employment in the industry will have changed as the result of changes in one or more of the other determinants of employment, to which must be added the change in wage rates themselves if we adopt a period *after* the change in wages.

Our practical consideration concerns the length of the period in which changes in employment affect wage rates. It is extremely doubtful if wages are ever an "offer price" under contemporary conditions. Certainly in the capital-goods industries, which are on the whole well organized in trade unions, one would expect the connection between wages and changed employment to be established in roughly the following order: increase in employment, recognition of strengthened bargaining power of union, pressure for wage advances. (Mr. Keynes is probably correct about the relative importance placed by trade unions upon real and money wages, but he appears to underestimate the importance of strategic considerations.) Even where effective trade-union organisation is absent, the order seems to be the same; increases in wages appear to result from increased employment rather than to be one of the conditions rendering it possible. Indeed the presence of relatively conservative trade unions, relying upon long-term agreements which may or may not include clauses relating wages to the cost of living, may be a factor making for stability of wages in face of rising employment.

By the light of these conclusions it appears that the most suitable method of measuring the elasticity of supply of labour in both of our groups is to take the average of the elasticities obtained by the comparison of June and October employment with December wages; this gives us for the capital-goods industries:

1932–4	1934–5	1935–6	1936–7
∞	4.43	2.16	2.12;

and for the consumers'-goods industries:

1932–4	1934–5	1935–6	1936–7
∞	0.03	1.47	1.54.

these results are extremely interesting as they appear to indicate that the position was growing increasingly stable, as far as the risk of inflation was concerned, right up to the eve of the recession, and that had steps been taken to check the rise in unemployment early enough no danger to the price level would have resulted. In the capital-goods industries, as one would have expected in view of the high proportion of skilled labour needing long training, the elasticity of supply of labour steadily declined. In the consumer's-goods industries, on the other hand, the elasticity of supply was very low at the commencement of recovery, probably owing to the fact that employment in the industries concerned had declined relatively little during the depression, but became progressively greater year by year as new labour entered and completed the relatively short period of training necessary. The position can be summarized by the use of the formula of the previous section.

	1932–4	1934–5	1935–6	1936–7
e/E	1.0	0.01	0.67	0.73.

VI

The previous section applied the conclusions of our greatly simplified working model to a very complex modern economy. It goes without saying that the argument cannot rest there: its conclusions are of necessity invalid. Can we, therefore, estimate what allowances have to be made when reasoning from one to the other, and can we, having discovered in what fields the discrepancies lie, select measurable quantities which will indicate the extent of the necessary readjustment? This is clearly a selective task; almost every way in which our model is simplified calls for a corresponding allowance to be made. But two major difficulties, to which our attention will be confined, appear to overshadow the rest.

Firstly, even if relative wage costs were truly representative of relative prime costs at the commencement of our period, and even if employment was then an adequate measurement of output, a period of six years of technical progress and active investment cannot have failed to alter the relative coefficients of production between a large number of the industries concerned. Capital equipment as well as labour will have flowed into the consumer's-goods industries; it will also have entered the capital-goods industries and to some extent replaced the dwindling supply of specialised labour, or assisted less skilled labour to do so. It is very improbable that these two influxes can be measured and compared, or that the extent to which they have altered output per man-hour can be accurately estimated. But still, at the end of the period of expansion, the elasticity of supply of labour in the consumer's-goods industries was increasing, thus indicating

that either the flow of new labour into these industries still continued, or that the inflow of complimentary capital was increasing output per head sufficiently to permit total output to expand faster than the bargaining power of labour. In fact, Mrs. Robinson's conditions were being fulfilled. It appears that e/E is after all a reliable indicator, subsuming under its terms most of the operative factors, if an appropriate method of measurement is adopted, and if it operates within a closed system.

This last condition immediately confronts us with our second, and most formidable, difficulty. Great Britain is very far from being a closed system, and both consumers'- and capital-goods industries contribute to her foreign trade. This blurs almost to obliteration the already far from distinct dividing line between the two groups; an increase in the output of the heavy industries, if the product is exported, contributes indirectly to the supply of consumers' goods. In this connection no alteration in the rate of interest can offset increases in costs. Rising wages will not, in this case, have inflationary effects, as the necessary condition that the domestic monetary authorities continue to maintain the level of "investment" cannot therefore apply: indeed the reverse will be the case. Thus the conditions of supply of imported consumers' goods come into the question, and decide it as far as it can be decided if we conclude that the internal position cannot be further condensed into verifiable terms. The greater the elasticity of supply of this class of imports, the further, in all probability, employment can expand without danger of inflation.

This can be roughly estimated in the following manner. Taking the totals of food imports into the United Kingdom, and measuring the elasticity of supply by the volume estimates and a price index constructed from the value estimates, we obtain the following figures:

1932–4	1934–5	1935–6	1937–7
0.26	−0.29	0.57	0.06.

Between 1932 and 1934 there was a fall in prices, coupled with a lesser fall in the volume imported. Between 1934 and 1935 the fall in imports continued, but prices commenced to rise: for the other two years both the quantities were positive.

In conclusion, therefore, it seems that we can maintain the opinions reached in the previous section, thus qualified, without falling into grave error. Consequently it would appear that "reflationary" operations are somewhat safer, even in an unclosed economy like Great Britain, than is sometimes supposed, and that, in her case at least, the most dangerous period was the inauguration of recovery.

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NOTE ON FRISCH'S DIAGONAL REGRESSION

By Charles W. Cobb

With the origin at the means of the variables, the diagonal regression for x_1 is, in three variables,

$$x_1 = \pm \frac{\sigma_1}{\sigma_2} \frac{k_{13}}{k_{23}} x_2 \pm \frac{\sigma_1}{\sigma_3} \frac{k_{12}}{k_{23}} x_3, \qquad k_{13} = \sqrt{1 - r_{13}^2}, \text{ etc.};$$

while, in the same notation, the partial-correlation coefficients are:

$$r_{12.3} = \frac{r_{12} - r_{13}r_{23}}{k_{13} \cdot k_{23}}, \qquad r_{13.2} = \frac{r_{13} - r_{12}r_{23}}{k_{12} \cdot k_{23}};$$

and the least-squares regression is:

$$x_1 = \frac{\sigma_1}{\sigma_2} \frac{r_{12} - r_{13}r_{23}}{k_{23}^2} x_2 + \frac{\sigma_1}{\sigma_3} \frac{r_{13} - r_{12}r_{23}}{k_{23}^2} x_3.$$

Hence: The diagonal-regression coefficient times the corresponding partial-correlation coefficient is numerically equal to the corresponding least-squares regression coefficient.[1,2] This illustrates the meaning of the diagonal-regression coefficients.

If the signs of the Frisch regression coefficients are taken to agree with those of the corresponding partial-correlation coefficients, the Frisch regression has the advantage that it supplies us with a *unique* relation which defines any one of the three variables in terms of the other two.[3] There is, however, one and only one exception to this rule. Taking two simple correlation coefficients positive without loss of generality, then, in case all three partial-correlation coefficients are positive,[4] it is possible to define x_1, say, explicitly in terms of x_2 and x_3 with positive coefficients, but it is not possible *by the same relation* to define x_2 explicitly in terms of x_1 and x_3 with positive coefficients.

In all other cases (taking two simple correlations positive, as stated) the Frisch regression plane is unique (Appendix A).

In the exceptional case we may resort to a *line* regression of the form

$$\frac{x_1}{\sigma_1} = \frac{x_2}{\sigma_2} = \frac{x_3}{\sigma_3},$$

in case a unique relation is desired. Deviations may then be measured perpendicular to this line.

[1] This is true for any number of variables.

[2] Or, $\alpha_{ij \cdot k} = \sqrt{\beta_{ij \cdot k} \div \beta_{ji \cdot k}}$, where α, β are diagonal and least-squares coefficients respectively.

[3] Evidently, the least-squares regression has not this advantage.

[4] If all three partials are positive, then all three simples are positive.

The relation of this line to the least-squares planes is determined as follows: Given the x_2, x_3 co-ordinates of any point on the line, the corresponding values of x_1 for the line and for the least-squares plane defining x_1, respectively, are related by the equation

$$\frac{x_1 \text{ (plane)}}{x_1 \text{ (line)}} = \frac{r_{12} + r_{13}}{1 + r_{23}}.$$

Under the given conditions (all simple correlations positive, all partials positive) the right-hand side of this equation is a positive number less than 1. (Appendix B).

The standard error s, say, perpendicular to the regression line, is given by the equation

$$\frac{S^2}{2} = \frac{\sigma_1^2\sigma_2^2(1 - r_{12}) + \sigma_1^2\sigma_3^2(1 - r_{13}) + \sigma_2^2\sigma_3^2(1 - r_{23})}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}.$$

Standard errors relative to the other regression loci of this paper (in desired directions) may be easily computed.

APPENDIX

(A)

Let us assume throughout, for clearness:

(1) no correlation is 0, or ± 1;

(2) r_{12} is $+$, and r_{13} is $+$.

Then let us assume, for the present, that r_{23} is $+$. With these assumptions the following algebraic signs for the partial correlations are *possible* (independent of any regression). Dropping the symbol r and writing only the subscripts, the partial correlations 12.3, 13.2, 23.1 may be, in order,

$$\begin{array}{ccc} + & + & + \\ + & + & - \\ + & - & + \\ - & + & + \end{array}$$

and there are no other possibilities. At least two partials must be $+$.

It is impossible, for example, that

$$\begin{aligned} &12.3 \text{ is } +,\ 12 > 13 \times 23; \\ &13.2 \text{ is } -,\ 13 < 12 \times 23; \\ &23.1 \text{ is } -,\ 23 < 12 \times 13; \end{aligned}$$

for this leads to the impossible relation

$$13 \times 23 < 12^2 \cdot 13 \times 23.$$

Assuming now that r_{23} is $-$, the partials may be (in the same order as before)

$$+ \quad + \quad -$$

and there is no other possibility.

It is (again) impossible, for example, that the partials have the signs

$$+ \quad - \quad -$$

for in that case, we should have the impossible relation

$$13 < 12 \times 23$$

where 12 and 13 are both $+$, and 23 is $-$.

Under the general assumptions (1) and (2), therefore, at least two partials are $+$.

Now if two *and only two* partials are $+$, the diagonal-regression equations represent the *same plane*. Thus, for example, if 12.3 and 13.2 are $+$, and 23.1 is $-$, these equations are:

$$x_1 = \frac{\sigma_1}{\sigma_2} \frac{k_{13}}{k_{23}} x_2 + \frac{\sigma_1}{\sigma_3} \frac{k_{12}}{k_{23}} x_3,$$

$$x_2 = -\frac{\sigma_2}{\sigma_3} \frac{k_{21}}{k_{31}} x_3 + \frac{\sigma_2}{\sigma_1} \frac{k_{23}}{k_{31}} x_1,$$

$$x_3 = +\frac{\sigma_3}{\sigma_1} \frac{k_{32}}{k_{12}} x_1 - \frac{\sigma_3}{\sigma_2} \frac{k_{31}}{k_{12}} x_2,$$

and represent the same relation among x_1, x_2, x_3.

If, however, *all three* partial correlations are positive, the algebraic signs of the diagonal equations are incompatible, and these equations can no longer represent the same relation among the variables (same plane).

(B)

When the ratios

$$\frac{x_1}{\sigma_1}, \quad \frac{x_2}{\sigma_2}, \quad \frac{x_3}{\sigma_3},$$

have the value λ, say, we have

$$x_2 = \lambda\sigma_2, \quad x_3 = \lambda\sigma_3,$$

and the corresponding value of x_1 on the *least-squares plane* defining x_1 is

$$x_1 = \lambda\sigma_1 \frac{(r_{12} + r_{13})}{1 + r_{23}} \cdot$$

The corresponding value of x_1 on the chosen regression *line* is

$$x_1 = \lambda\sigma_1.$$

Therefore

$$\frac{x_1\,(\text{plane})}{x_1\,(\text{line})} = \frac{r_{12} + r_{13}}{1 + r_{23}}\,.$$

When all the simple correlations are positive and all the partials are positive, this ratio is a positive number less than 1, as follows:

Given

$$\begin{aligned} 12 &> 13 \times 23, \\ 13 &> 12 \times 23, \\ 23 &> 12 \times 13, \end{aligned}$$

the proposition is evidently true if 23 is equal to or greater than 12 or 13.

Suppose then

$$23 < 13 < 12.$$

By hypothesis

$$1 + 23 > 1 + 12 \times 13,$$

and in turn

$$1 + 12 \times 13 > 12 + 13,$$

for

$$1 + 12 \times 13 \equiv 12 + 13 + (1 - 12)(1 - 13).$$

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COMPTE RENDU DE LA RÉUNION D'ANNECY 12–15 SEPTEMBRE 1937

Par GEORGES LUTFALLA

LA SEPTIÈME réunion européenne de la Sociéte internationale d'Économétrie s'est tenue à Annecy, du 12 au 15 septembre 1937. Les détails matériels d'organisation minutieusement arrêtés par M. Tinbergen, ainsi que la liste des participants ont déja été rapportés dans une précédente communication.[1]

Les séances de la journée d'ouverture furent présidées par le Professeur Bowley, alors Vice-Président en exercice de la Société. Elles furent consacrées à des exposés d'ensemble sur les mathématiques et la statistique dans leur relation avec la théorie économique: MM. R. W. James, T. Koopmans, et P. de Wolff y prirent successivement la parole.

M. R. W. JAMES (Melbourne) a traité de *l'importance des solutions caractéristiques des équations différentielles et aux différences finies* qui se présentent en dynamique économique. Cet exposé est déjà publié.[2]

M. T. KOOPMANS (Rotterdam) a fait un exposé systématique des résultats publiés dans sa thèse sur les *analyses de régression linéaire des séries économiques temporelles*. Le problème est connu: soient un nombre déterminé de variables X_k ($k=1, 2, \cdots, K$) et les valeurs observées X_k^t au cours de périodes successives $t=1, 2, \cdots, T$. Si des raisons théoriques permettent de songer à une relation linéaire entre les variables X_k, et, si des erreurs entâchent les relevés: estimer les coefficients de la relation linéaire et se former une opinion sur la valeur de ces estimations. Ces erreurs, dans les problèmes économiques, peuvent appartenir à l'un des deux types suivants: a) erreur d'observations; b) erreur provenant de ce que l'on néglige des variables peu importantes. M. Koopmans insista tout particulièrement sur le caractère synthétique de sa théorie, en commentant le tableau No. 1 (les β et γ y représentent des coefficients de régression).

M. DE WOLFF (La Haye) a également considéré un système économique dont l'une des variables vérifie la relation:

$$\frac{dy}{dt} = ay(t) - by(t-1), \tag{1.1}$$

donnée dans l'exposé de M. James.

Il s'est posé la question de savoir quels seront *les mouvements résultantis lorsque le système économique caractérisé par (1.1) est perturbé par une*

[1] ECONOMETRICA, Vol. 6, janvier 1938, p. 91.
[2] ECONOMETRICA, Vol. 6, octobre 1938, pp. 326–343.

TABLEAU No. 1

COMPARAISONS DE DIVERSES THEORIES DE LA REGRESSION

	I—THÉORIE NORMALE (K. Pearson et Rider)	II—THÉORIE DE FISHER	III—THÉORIE DE FRISCH	IV—SYNTHÈSE DES THÉORIES II ET III
Méthode.	Théorie des épreuves.	Théorie des épreuves: variable dépendante privilégiée.	Pas d'épreuves. Toutes les hypothèses formulées concernent les observations. Pas de variables privilégiées.	Théorie des épreuves utilisant des paramètres inconnus qui ne peuvent être estimés.
Hypothèses.	$X_k (k=1, 2, \cdots, K)$ aléatoires tirés d'une distribution normale.	$X_2, X_3, \cdots, X_K$ ne changent pas au cours d'épreuves répétées. $X_1 = \xi_1 + \zeta_1$, $\epsilon_1 = \beta_{12}X_2 + \cdots + \beta_{1K}X_K$, ζ_1 aléatoire tiré d'une distribution normale de moyenne nulle et de variance inconnue σ.	$X_k = \xi_k + \zeta_k (k=1, 2, \cdots, K)$, $\xi_1 = \gamma_{12}\xi_2 + \cdots + \gamma_{1K}\xi_K$, ζ sans corrélation avec ξ_k, ξ_h et ζ_h $(h \neq k)$. Rapports $\epsilon_1 : \epsilon_2 : \cdots : \epsilon_K$ des variances des ζ_k inconnus.	$X_k = \xi_k + \zeta_k$, $\xi_1 = \gamma_{12}\xi_2 + \cdots + \gamma_{1K}\xi_K$, ζ_k aléatoire tiré de distribution normale de moyenne nulle et de variance inconnue $\sigma\epsilon_k$. Rapports $\epsilon_1 : \epsilon_2 : \cdots : \epsilon_K$ des variances des ζ_k inconnus.
Coefficients de régression estimés.	K séries de régressions élémentaires.	Coefficients de première régression élémentaire.	Coefficients de régression diagonale.	Coefficients de régression pondérés, supposant donnés les rapports $e_1 : e_2 : \cdots : e_K$.
Conditions du résultat.			Cas de colinéarité multiple approchée exclus.	Dans ces cas, les approximations sont insuffisantes.
Forme des résultats.	Distribution par épreuves des coefficients de régression.	Coefficients de régression suivant distribution normale. Ceux-ci ne sont pas entachés d'erreurs accidentelles (*unbiased*) leurs variances dépendent de σ et des valeurs de $X_2, \cdots, X_K$.	La "régression vraie" (pour $K=2$) est comprise entre les régressions élémentaires. La régression diagonale est adoptée comme estimation, le déploiement des régressions élémentaires comme indication de leur exactitude.	*Erreur d'épreuves*. Approximativement distribution normale des coefficients de régression. Les variances dépendent de σ, les ϵ_k et les valeurs de $\xi_k, \cdots$. *Erreur de pondérations* des moyennes de ϵ_k et e_k. Tant que la régression pondérée est *biased*, la régression correctement pondérée est comprise dans l'angle spatial des régressions élémentaires. On peut restreindre cette limite en supposant que les variances estimées se_k de ζ_k ne sont pas trop grandes.

force extérieure cyclique $F(t)$, si les réactions sur le système perturbateur sont négligeables. Il faut alors d'ajouter $F(t)$ au deuxième membre de (1.1). En fait, il suffit d'étudier le cas de F périodique, s'il n'en est pas ainsi un développement en série de Fourier y ramène. Dans ce cas

$$Rf = F(t) = a \cos \lambda t + b \sin \lambda t, \tag{3.1}$$

a et b sont des constantes et R un opérateur linéaire portant sur les opérateurs élémentaires D et Δ, ($Df = df/dt$, $\Delta f = f_{-1}$). Il est aisé de montrer qu'il est toujours possible de trouver une *solution particulière* de (3.1) de même forme que $F(t)$, mais avec des constantes différentes. La démonstration qui n'a été donnée que pour l'opérateur Δ [(3.1) se réduit alors à $\sum c_k \Delta x_i^k = 0$] est fondée sur les identités suivantes

$$\begin{aligned}\Delta(p \cos \lambda t + q \sin \lambda t) &= p \cos (\lambda t - \lambda) + q \sin (\lambda t - \lambda)\\ &= (p \cos \lambda - q \sin \lambda) \cos \lambda t\\ &\quad + (p \sin \lambda + q \cos \lambda) \sin \lambda t\\ &= p' \cos \lambda t + q' \sin \lambda t = (p', q') = (p, q)D_\lambda.\end{aligned}$$

Par itération

$$\Delta^k(p, q) = (p, q)(D_\lambda)^k = (p, q)D_{k\lambda}.$$

Avec nos notations

$$(a, b) = \sum c_k \Delta^k(p, q) = (p, q) \sum c_k \Delta_{k\lambda} = (p, q)M, \tag{3.2}$$

p et q peuvent être tirés des équations (3.2), si $M \neq 0$, c'est-à-dire lorsque

$$\begin{aligned}\left\| \sum c_k D_{k\lambda} \right\| &= \left(\sum c_k \cos k\lambda\right)^2 + \left(\sum c_k \sin k\lambda\right)^2\\ &= \sum c_k c_l \cos (k - l)\lambda \neq 0.\end{aligned}$$

Si $M = 0$, (3.1) admet une solution du type $t(p \cos \lambda t + q \sin \lambda t)$. Si $M \neq 0$, l'amplitude $A = \sqrt{a^2 + b^2}$ du mouvement perturbateur et l'amplitude $Q = \sqrt{p^2 + q^2}$ du mouvement résultant sont liées par le relation simple

$$Q = \frac{A}{\sqrt{|M|}};$$

il n'y a pas de changement de période. La solution générale de (3.1) peut toujours être mise sous la forme de la solution générale de $Rf = 0$, à laquelle on ajoute une solution particulière de (3.1). Nous venons de trouver une telle solution particulière et nous avons vu que la période est égale à la période de la force perturbatrice. Il résulte de ces considérations que des mesures prises à l'intérieur du système ne peuvent modifier la période. Seule l'amplitude peut être changée en vue de contrecarrer les perturbations, en rendant M aussi grand que possible.

Pour illustrer ces résultats, M. de Wolff a imprimé au système considéré par M. Tinbergen, dans un article récent,[3] une force externe du type $F(t)=\sin \pi t/4$.

La journée du 13 septembre fut consacrée à la théorie de la consommation. Sous la présidence de M. G. Lutfalla, les exposés ci-après furent discutés.

PROFESSEUR A. L. BOWLEY (Londres): *Calcul numérique de l'élasticité de substitution.* Soient (x_i, p_i, e) et $(x_i+dx_i, p_i+dp_i, e+de)$ les prix, les quantités d'un bien déterminé $[i]$, $(i=1, 2, \cdots, n)$ et la dépense totale d'un individu-type à deux moments différents. Les quantités suivantes dérivent des précédentes.

$$e_i = x_ip_i = w_ie, \qquad \sum e_i = e, \qquad \sum w_i = 1, \qquad dp_i = r_ip_i.$$

La variation relative des prix sera calculée à partir d'un indice de Laspeyres par la quantité ρ définie par

$$\frac{\sum x_i(p_i + dp_i)}{\sum x_ip_i} = 1 + \rho,$$

d'où $\rho e=\sum x_ir_ip_i$, et enfin $\rho=\sum w_ir_i$. Soit

$$v=de/e-\rho, \tag{4.1}$$

excès du revenue sur l'accroissement relatif du prix. Les goûts de l'individu-type sont caractérisés par une fonction indice d'utilité supposée constante au cours de la période considérée $\phi(x_i, x_2, \cdots, x_n)=C^{te}$. Soient encore ϕ_i la dérivée première par rapport à x_i, ϕ_{ij} la dérivée seconde par rapport à x_i et x_j, et Φ le déterminant constitué par les ϕ_{ij} bordés par les ϕ_i et enfin Φ_i, Φ_{ij} les mineurs correspondant à ϕ_i et ϕ_{ij}. Ces notations posées, au *point d'équilibre* les utilités pondérées sont égales

$$\frac{1}{p_1}\phi_1 = \frac{1}{p_2}\phi_2 = \cdots = \frac{1}{p_n}\phi_n = \lambda. \tag{4.2}$$

Par dérivation et calculs sans difficulté, on obtient

$$\frac{d\lambda}{\lambda}\phi_i + r_i\phi_i = \phi_{i1}dx_1 + \phi_{i2}dx_2 + \cdots + \phi_{in}dx_n, \tag{4.3}$$

$$(i = 1, 2, \cdots, n).$$

De (4.1) et de (4.2), on tire pareillement

$$\lambda ev = \phi_1dx_1 + \phi_2dx_2 + \cdots + \phi_ndx_n. \tag{4.4}$$

[3] *De Economist*, 1937, II, p. 81.

Le système de $n+1$ équations linéaires (4.3)+(4.4) est suffisant pour tirer les valeurs de $dx_1, dx_2, \cdots, dx_n, d\lambda/\lambda$ à savoir:

$$\Phi dx_i = \lambda ev\Phi_i + \sum_j r_j\phi_j\Phi_{ij} = \lambda ev + \sum_j (r_j - r_i)\phi_j\Phi_{ij}, \tag{4.5}$$

$$\Phi\frac{d\lambda}{\lambda} = -\lambda ev\Phi_0 - \sum r_i\phi_i\Phi_i. \tag{4.6}$$

Dans *Family Expenditure*, on a montré qu'*avec des prix constants* la dépense moyenne de [i] est donnée par $e_i = k_ie + c_i$, $\sum k_i = 1$, où $k_i = \phi_i\Phi_i/\Phi$. Il y est pareillement indiqué que l'élasticité de la dépense en [i] par rapport au revenu s'exprime par $\eta_i = k_i/w_i$ et que l'*élasticité partielle de substitution* de x_i en x_j qui se définit par $\sigma_{ij} = d\log(x_i:x_j)/d\log(\phi_i:\phi_j)$ s'exprime, au point d'équilibre, par $\sigma_{ij} = \lambda e/x_i:x_j\cdot\Phi_{ij}/\Phi$. Les résultats (4.5) et (4.6) peuvent dès lors être modifiés:

$$p_idx_i = evk_i + e\sum w_iw_j(r_j - r_i)\sigma_{ij}, \tag{4.7}$$

$$\frac{d\lambda}{\lambda} = -\lambda ev\frac{\Phi_0}{\Phi} - \sum r_ik_i. \tag{4.8}$$

Les k_i étant déterminés à partir d'une série de budgets de base, si les variations des prix et des dépenses sont connues, les σ_{ij} demeurent les seules inconnues. Pour chaque année, il y a $n-1$ équations indépendantes (puisqu'en les ajoutant membre à membre, on aboutit à une identité). Si les données précédentes sont fournies au cours de m années différentes, on a $m(n-1)$ équations pour déterminer $n(n-1):2$ quantités. Pour $n=4$, $m=2$, les calculs sont faisables; pour $n>4$, ils deviennent très pénibles. Un cas particulier est intéressant: $n=2$ et $m=1$. L'expression de l'élasticité de substitution se réduit à

$$\sigma_{12} = \frac{\eta_2\dfrac{dx_1}{x_1} - \eta_1\dfrac{dx_2}{x_2}}{r_2 - r_1} = \frac{k_2p_1dx_1 - k_1p_2dx_2}{w_1x_2dp_2 - w_2x_1dp_1}, \tag{4.9}$$

(formule valable même si le produit $dp_i\cdot dx_i$ n'est pas négligeable comme on l'a supposé plus haut), où $k_1+k_2=1$, $w_1+w_2=1$. Si, au lieu de considérer isolément deux biens [i] et [j], on compose l'ensemble des biens existants en deux groupes, les biens nécessaires [1] et les biens de luxe [2], la formule (4.9) est encore valable, à condition de définir convenablement les prix.

M. Gibrat fit remarquer qu'à la base de cet exposé se trouve l'hypothèse de la constance de la fonction ϕ. Elle est contestable: ayant eu l'occasion d'étudier l'action de changements brusques des prix dans certains services publics (transports), il a remarqué que la constance n'était certainement pas respectée. M. Wiśniewski a fait des constatations analogues à propos des téléphones polonais.

Dans une deuxième communication, adoptant les mêmes notations, M. Bowley s'est proposé de rechercher *la condition pour que l'indice de Laspeyres* $P_0=\sum x_i(p_i+dp_i)/\sum x_ip_i$ *soit plus grand que l'indice de Paasche* $P_1=\sum(x_i+dx_i)(p_1+dp_i)/\sum(x_i+dx_i)p_i$. De ces définitions, on tire immédiatement

$$(5.1)\qquad (P_0 - P_1)\sum (x_i + dx_i)\, p_i = \rho\sum p_idx_i - \sum dx_idp_i.$$

En posant encore $v=de/e-\rho$, (5.1) devient

$$(5.2)\qquad (P_0 - P_1)\sum (x_i + dx_i)p_i = \rho ev - P_0\sum r_i^2\eta_{ii},$$

où $\eta_{ii}=p_i/x_i\cdot dx_i/dp_i$ devient l'élasticité de la demande de la marchandise $[i]$ par rapport à son propre prix, lorsque dx_i et dp_i deviennent petits. De (5.2), il suit: 1°) si tous les $\eta_{ii}<0$, $P_0>P_1$ si $\rho v>0$, inégalité vérifiée avec des prix croissants, lorsque la dépense croît plus rapidement que les prix, ou, avec des prix décroissants, lorsque la dépense décroît plus rapidement que les prix. Si $v=0$, ou si $\rho=0$ (mais pas tous les r_i), alors $P_0>P_1$. Mais si ρ et v sont de signes opposés, $P_0<P_1$, à moins que les η_{ii} soient très grands; 2°) si les $\eta_{ii}>0$, comme cela peut être le cas lorsque dx_i, dp_i, et v ne sont pas petits, P_0 peut être moindre que P_1.

Dans sa communication sur les *indices économiques et la disparité*, M. Dumay (Paris) s'est attaché à relier certains indices connus à la définition différentielle de M. Divisia $dI/I=\sum x_idp_i/\sum x_ip_i$. A cet effet, il définit un système économique déterminé par l'ensemble des prix et des quantités de n marchandises $(p_1, p_2, \cdots, p_n, x_1, x_2, \cdots, x_n) = (\mathbf{x}_i\mathbf{p}_i)$. Au cours du temps ou en passant d'un lieu à un autre, ces grandeurs varient: leurs variations caractérisent les transformations subies par le système considéré. Deux états $(\mathbf{x}_i, \mathbf{P}_i)$ et $(\mathbf{x}'_i, \mathbf{P}_i')$ sont semblables, lorsqu'il est possible de faire correspondre à chaque élément de l'un, un élément de l'autre, de telle façon que, pour passer d'un système à l'autre, il suffise de multiplier tous les prix par un certain facteur, et toutes les quantités par un autre facteur $(\mathbf{x}_i', \mathbf{P}_i') = (a\mathbf{x}_i, b\mathbf{P}_i)$. Il n'en est pas ainsi en général, les états considérés sont *dissemblables*.

Il est bien évident que l'on peut passer de l'état $(\mathbf{x}_i, \mathbf{P}_i)$ à un état $(\mathbf{x}_i', \mathbf{P}_i')$ par une infinité de transformations. A chaque transformation correspond une valeur de I, de telle sorte que pour deux transformations différentes correspondent en général deux indices I_1 et I_2, tels que $\Delta=I_2-I_1\neq 0$. Parmi toutes les transformations imaginables, deux familles sont particulièrement intéressantes: les *dilatations* (ou *contractions*) pour lesquelles la variation relative du prix de chaque marchandise est égale à la variation relative de sa quantité $dp_i/p_i=dx_i/x_i$ (dans le plan Op_ix_i: mouvement le long d'une droite $x_i=hp_i$ passant

par l'origine); et les *glissements* pour lesquels la variation relative du prix de chaque marchandise est égale et de signe contraire à la variation relative de sa quantité $dp_i/p_i = -dx_i/x_i$ (dans le plan Op_ix_i, mouvement le long d'une hyperbole équilatère $p_ix_i = k$, admettant les axes comme asymptotes). Par un état économique donné, il passe n infinités de dilatations et n infinités de glissements. Si l'on considère les n infinités de dilatations D_1 passant par $(\mathbf{x}_i, \mathbf{P}_i)$ et les n infinités de glissements G_1 passant par $(\mathbf{x}_i', \mathbf{P}_i')$, ces deux ensembles de n infinités de transformations ont n points communs définissant un troisième état économique $(\mathbf{x}_i'', \mathbf{P}_i'')$ caractérisé par le fait que le produit p_ix_i est le même pour le deuxième état, et le rapport p_i/x_i est le même que pour le premier. De même, on peut considérer les n infinités de glissements G_2 passant par $(\mathbf{x}_i, \mathbf{P}_i)$ et les n infinités de dilatations D_2 passant par $(\mathbf{x}_i', \mathbf{P}_i')$. On peut définir d'une façon univoque un indice de dissemblance Δ par la différence des variations de I résultant des transformations D_1+G_1 et G_2+D_2. Soit M'' le point commun du glissement G_2 et de la dilatation, D_2 et M''' le point commun de la dilatation D_1 et du glissement G_1. Désignons par (π, ξ) les coordonnées courantes d'un point quelconque du contour $MM''M'M'''$ ainsi défini. Sur un diagramme doublement logarithmique, ce contour se transforme en un rectangle dont les côtés sont parallèles aux bissectrices des axes. Si l'on fait varier les prix et les quantités en fonction d'un seul paramètre t, en supposant que le point figuratif parcourt chacun des côtés du rectangle dans l'unité de temps, les coordonnées d'un point quelconque sont données par (en supprimant l'indice i):

$$MM''\begin{cases} L\pi = Lp + \alpha t, \\ L\xi = Lx - \alpha t, \end{cases} \qquad M''M'\begin{cases} L\bar{\omega} = Lp' + \beta(t-1), \\ L\xi = Lx' + \beta(t-1), \end{cases}$$

ou $\alpha = \dfrac{1}{2} L \dfrac{xp'}{x'p} = \dfrac{1}{2} L \dfrac{\rho'}{\rho}$ et $\beta = \dfrac{1}{2} L \dfrac{x'p'}{xp} = \dfrac{1}{2} L \dfrac{r'}{r}$. En intégrant le long de MM'' et de $M''M'$, il vient

$$L\frac{I''}{I} = \frac{1}{2}\frac{\sum rL(\rho':\rho)}{\sum r} \quad \text{et} \quad \frac{I''}{I'} = \sqrt{\frac{\sum r}{\sum r'}}.$$

On trouverait de même si l'on intégrait le long du contour $MM'''M'$

$$\frac{I'''}{I} = \sqrt{\frac{\sum r'}{\sum r}} \quad \text{et} \quad L\frac{I'}{I'''} = \frac{\sum r'L(\rho':\rho)}{\sum r'}.$$

Deux dilatations donnent donc la même variation de l'indice: seuls les glissements sont générateurs de dissemblance. En l'occurrence la dissemblance peut être définie par

$$\Delta = \frac{\sum r'L(\rho':\rho)}{\sum r'} - \frac{\sum rL(\rho':\rho)}{\sum r}.$$

Une abaque permet un calcul rapide de Δ.

L'exposé de M. Demay fut suivi d'une communication de M. WALD (*p.t.* Genève) sur la *théorie générale des indices;* cette communication est aujourd'hui publiée.[4]

Dans sa communication sur la *détermination des unités de consommation,* M. JAN WIŚNIEWSKI (Varsovie) a proposé une méthode permettant d'écarter les considérations physiologiques qui encombrent la question. Il s'agit de trouver la proportion des revenus monétaires qui peuvent être considérés comme équivalents, eu égard à l'importance et la composition des familles. Des revenus sont équivalents s'ils engendrent une égale satisfaction: cette grandeur n'étant pas mesurable, on peut adopter, comme critère d'équivalence, le pourcentage de la dépense totale affecté à l'acquisition des aliments, c'est ce que l'on appellera le "rapport alimentaire." Si ce rapport y est lié au revenu relatif à l'unité consommatrice par une équation de régression linaire, on a

$$\begin{aligned} y &= c - b\frac{x}{A_1n_1 + A_2n_2 + \cdots + A_kn_k} \\ &= c - \frac{x}{a_1n_1 + a_2n_2 + \cdots + a_kn_k}, \end{aligned} \tag{7.1}$$

où x est la dépense totale, $n_1, n_2, \cdots, n_k$, le nombre de personnes constituant chaque classe d'âge ou de sexe, $A_1, A_2, \cdots, A_k$ les fractions de l'unité de consommation propres à chacune des classes ci-dessus. Les constantes c et b sont déterminées par la méthode des moindres carrés grâce au procédé d'approximation indiqué par D. Brundt,[5] après avoir pris l'un des $A_k = 1$.

Après quelques remarques du PROFESSEUR BOWLEY concernant la corrélation entre numérateur et dénominateur dans (7.1), MLLE. JOSEPH présenta une brève analyse de la méthode employée aux U.S.A. pour la détermination des unités par des données physiologiques. M. GIBRAT à son tour fit valoir la simplification des calculs qui pourraient résulter d'équations telles que

$$\begin{aligned} y &= c + b\frac{A_1n_1 + \cdots + A_kn_k}{x}, \\ y &= c + ax + A_1n_1 + \cdots + A_kn_k. \end{aligned} \tag{7.2}$$

[4] "Zur Theorie der Preisindexziffern," *Zeitschrift für Nationalökonomie,* Band VIII, Heft 2, 1937, pp. 179–219.

[5] *Combination of Observations,* 2nd éd., pp. 75–76.

Des données relatives à l'industrie cotonnière ont permis à M. VICTOR EDELBERG (Londres) de développer une intéressante théorie de l'*hystérésis de la demande.* Le coton transformé dans le Lancashire appartient à une variété de courte fibre assez répandue: cette circonstance permet aux filateurs de substituer, le cas échéant, au coton américain, qui est le coton le plus employé, le coton indien ou le coton brésilien. Soient $A(t)$ la demande du Lancashire en coton américain par rapport à l'ensemble du coton consommé (ce que nous appellerons la "demande" du coton américain), et p l'écart entre le prix du coton américain p_a et du coton indien p_i, $p=p_a-p_i$ (ce que nous appellerons le "prix" du coton américain).

Le point (A, p) décrit dans le plan OAp, entre 1929 et 1937, deux boucles d'une spirale et en amorce une troisième. Deux ordres d'explication: (1) La majeure partie des livraisons est constituée par des quantités achetées six mois auparavant. La demande courante $A(t)$ dépend dès lors des prix les plus récents $p(t)$, aussi bien que du prix courant $p(t)$. (2) La résistance qu'opposent les filateurs à substituer une variété à une autre. Cette résistance provient des délais nécessaires au réajustement des machines et aux délais exigés pour des livraisons d'une nouvelle variété. On peut dès lors dire que la demande $A(t)$ en 1937 a été faible parce qu'elle avait été déjà faible entre 1929 et 1937. Cette spirale descendante est donc le résultat de ce que l'on pourrait appeler un hystérésis tendanciel.

Si $p(\tau)$ exécute plusieurs cycles complets entre τ_1 et τ_2 $[p(\tau_1)=p(\tau_2)]$, l'intégrale $\int_{\tau_1}^{\tau_2} A(\tau)p'(\tau)d\tau$ représente l'*effet dissipatif* résultant des changements de prix. D'où analogie intéressante avec les cycles magnétiques. La formule générale d'hystérésis de la demande est

$$A(t) = F\left[p(t),\ \overset{t-1}{\underset{t-\sigma}{p(\tau)}},\ \overset{t-\sigma}{\underset{t-\omega}{A(\tau)}}\right]. \tag{8.1}$$

Si le mois est pris pour unité, σ est approximativement égale à 6 et $t-\omega$ est la date la plus éloignée dans le passé que l'on considère. Dans la formule $\partial F/\partial p(t)<0$, la dérivée fonctionnelle par rapport à $p(\tau)$, au point $\tau=\xi_1$ est $F'_{p(\xi_1)}<0$. La dérivée fonctionnelle par rapport à $A(\tau)$, au point $\tau=\xi_2$, est $F'_{A(\xi_2)}>0$. Dans les données, on relève une forte présomption de complémentarité intertemporelle. La sensibilité de la demande courante par rapport au prix croît avec l'importance de la demande passée, c'est-à-dire que la dérivée fonctionnelle de $\partial F/\partial p(t)$ par rapport à $A(\tau)$, $(\partial F/\partial p(t))'_{A(\xi_2)}$ est <0 et la deuxième dérivée $F''_{A(\xi_2)A(\xi_3)}$ est >0, pour $\xi_2<\xi_3$, etc.

On doit en conclure qu'une équation intégrale linéaire est trop restrictive pour expliciter (8.1) On peut lui substituer une formule plus simple analogue à

$$A(t) = C[p(t)]^{\alpha}[\bar{p}]^{\beta}A_r^{\gamma_r} \tag{8.2}$$

où C et γ_r sont des constantes positives, α et β des constantes négatives; $\bar{p}$ est la moyenne des prix récents et A_r est la demande moyenne au cours de r intervalles passés.

Sous la présidence de M. Staehle, la troisième journée a été occupée par des communications sur divers thèmes: productivité du travail, prévisions agricoles, et questions monétaires.

L'intérêt de l'exposé de M. Tinbergen (*p.t.* Genève) réside non seulement dans le sujet traité, mais aussi dans l'extension de sa méthode appliquée jusqu'ici aux phénomènes cycliques. Recherchant l'*influence des variations de la productivité du travail sur le mouvement de longue durée de l'emploi,* il a délibérément utilisé un système de 17 équations comprenant des relations logiques et des relations tirées de l'expérience des Pays-Bas (1923–1933). Dans ce résumé, il est impossible de faire une énumération des 33 grandeurs introduites dans l'analyse. Indiquons seulement qu'il considère comme inconnus: l'emploi total a, la production des biens de consommation u, celle des capitaux v, leur prix respectif p et q, la production des capitaux d'extension $\dot{C}$ et enfin les revenus autres que ceux du travail $\dot{E}$. Si l'on suppose que ces variables sont liées par une relation linéaire aux salaires (variable d'action) l, au travail par unité de biens de consommation g et de biens capitaux g', à la quantité de capitaux par unité de production h, à la quantité de monnaie en circulation M et enfin les exportations u'_A, les coefficients de regression trouvés sont donnés dans le tableau No. 2.

Tableau No. 2

	$\dot{l}$	$\dot{g}$	$\dot{g}'$	$\dot{h}$	M	u_A'	C^{te}
$\dot{a}$	0,09	3,50	0,66	−0,96	1,26	0,40	0,013
$\dot{u}$	0,58	1,45	0,64	−4,60	2,14	1,92	0,021
$\dot{v}$	−0,14	−0,41	−0,11	−0,39	2,09	−0,16	0,023
$\dot{u}'$	0,58	1,45	0,64	−4,60	2,14	0,92	0,021
$\dot{C}$	−0,06	−0,16	−0,04	0,16	0,84	−0,06	0,009
$\dot{p}$	0,27	1,00	0,08	1,00	—	—	—
$\dot{q}$	0,36	—	1,00	—	—	—	—
$\dot{E}$	0,20	0,48	0,22	−1,68	1,16	0,70	0,012

On en conclut qu'à la longue toute augmentation de la productivité du travail est désavantageuse à un accroissement de l'emploi (de même qu'à brève échéance, les travailleurs ont à pâtir des inventions). Ces résultats feront d'ailleurs l'objet d'une publication spéciale prochaine sous une forme encore plus élaborée qu'au moment de leur présentation.

Dans sa communication M. van der Schalk (Bussum) s'est proposé de rechercher l'*influence de la durée d'emploi sur la productivité du travail*

et plus spécialement l'action sur ce facteur de la loi de 8 heures. Son analyse statistique a porté sur des branches très variées de l'industrie hollandaise (mines, minoteries, brasseries, beurreries, féculeries de pommes de terre, sucre, cacao, distileries, vinaigreries, huileries, papeteries, ferblanteries, carrières de pierres, bâtiments, constructions de navires). Aucune corrélation immédiate n'est décelable; cela résulterait de la présence de divers facteurs: phases variées de la conjoncture, inégalité plus ou moins forte des salaires; ou à certains facteurs spéciaux aux industries considérées, comme le tonnage variable des navires. Abandonnant cette première tentative, M. van der Schalk a recherché s'il existait une relation avec le mouvement de longue durée de la productivité, par l'utilisation des *bunch maps*. Les résultats dans le tableau No. 3 sont dès lors plus significatifs:

TABLEAU No. 3

R	0,6–0,7	0,7–0,8	0,8–0,85	0,85–0,90	0,90–0,95	0,95–1,00
nombre d'industries	3	1	1	3	3	7

L'influence est négative dans les industries exigeant beaucoup de main d'oeuvre (carrières, ferblanteries, brasseries, bâtiments); elle est négligeable pour les industries exigeant une part relativement considérable de capital (constructions de navires, mines); elle est négligeable dans certains cas particuliers (beurreries, féculeries).

Cette communication a donné naissance à une longue discussion. M. Bowley a demandé notamment si l'augmentation constatée de la productivité, dans certains cas, admet un maximum. On pourrait évidemment l'admettre, puisque *a priori* il y a une baisse de la productivité dès que la durée est très restreinte. Un intéressant échange de vues de MM. T. Koopmans, Edelberg a immédiatement suivi sur les conditions de l'expérience de M. van der Schalk: distinction entre la durée et la tendance, inclusion des salaires, modification de l'équipement (dont il n'a pas été tenu compte dans les recherches présentes).

M. Fleming (Genève) a présenté une application de la corrélation multiple à la *recherche de la quantité de monnaie détenue par les sujets* d'une nation. Il s'agissait en l'occurrence de trouver l'une des équations d'un modèle d'étude du cycle du mouvement des affaires. Après une discussion systématique des facteurs, M. Fleming retient le taux de l'intérêt, le revenu national, la profitabilité des investissements (capitaux réels), la valeur du stock des capitaux, la distribution du capital et du revenu. Quelques-uns ou tous ces facteurs agissent avec certains décalages, sans que sur les données annuelles ceux-ci soient décelables. Des calculs fondés sur des statistiques anglaises ou améri-

caines d'après-guerre montrent que le taux de l'intérêt et le flux du revenu monétaire exercent une influence sur la demande de monnaie; la première négativement, la seconde positivement. De plus, on distingue l'existence d'autres actions tendancielles ou cycliques, sans qu'il soit possible de déterminer l'effet de chacun des autres facteurs ci-dessus énumérés. L'inverse du taux de l'intérêt, on le sait, a une action plus marquée que le taux lui-même: ce qui signifie qu'un taux d'intérêt bas agit plus sur la demande de monnaie qu'un taux élevé. Cette remarque amène à rechercher comment dans ces conditions la demande de monnaie s'ajuste à l'offre. Il semble que l'offre de monnaie réagisse, par l'intermédiaire du taux de l'intérêt, sur la dépense en capitaux et de la sorte sur le revenu monétaire et la demande de monnaie.

M. Samson (Genève) prenant le dernier la parole sur la *prévison des récoltes, méthodes et résultats* ouvrit une large discussion sur l'état présent de la question. Il convient tout d'abord de distinguer entre prévisions à longue et à courte échéance. Deux méthodes ont été jusqu'ici utilisées pour les *prévisions à longue échéance:* (1) la connaissance des cycles météorologiques. Ils apparaissent encore plus complexes que les cycles économiques, la détermination des longues périodes se heurte au manque de données sûres. On a voulu y suppléer par l'étude du niveau des lacs ou de la croissance des arbres californiens. Ces tentatives sont abandonnées aujourd'hui, au même titre que celles de Stanley Jevons (retour périodique des taches solaires) et celles de H. L. Moore (révolution de Vénus). Les seules périodes que l'on puisse retenir sont l'une de 35 ans (Brückner: très nombreuses séries; R. A. Fisher: pluie à Rothamsted; Samson: vin en France, 1849–1936), l'autre de 20 ans (Sir Napier Shaw; Weickmann et Gösele). (2) la liaison des récoltes à certains types météorologiques. La tentative la plus intéressante est celle de T. Okada: la récolte du riz au Japon dépend de l'insolation d'août; or la température de ce mois au Japon est liée à la pression barométrique de mars à mai à Santiago du Chili.

Les *prévisions à brève échéance* sont obtenues par deux procédés, qui loin de s'exclure peuvent être utilisées conjointement: (1) Estimation des récoltes au cours de leur croissance par des fermiers ou des autorités rurales. Les indications mensuelles publiées sont décevantes. Un rapprochement systématique des prévisions avec les résultats effectifs amène aux conclusions suivantes: à l'exception des prévisions qui précèdent de très près la récolte, les conjectures sont mauvaises. On peut ajouter que les prévisions en matière de céréales sont moins bonnes que celles concernant la betterave par exemple. La valeur de la prévision ne croît pas toujours avec l'approche de la récolte: meilleurs renseignements pour le blé d'hiver aux U.S.A. en décembre qu'en avril ou mai. On peut néanmoins tenir compte de certaines erreurs systéma-

tiques, telles que la surestimation des dommages ou la sous-estimation de l'intensité des variations; l'*U.S. Crop and Livestock Reporting Service* procède de la sorte. Pareillement M. Samson a-t-il pu améliorer les annonces faites pour le blé de printemps (U.S.A. $r^2=0.85$; Canada, 0.78) et le vin (Allemagne et Suisse, $r^2=0.87$). Cette méthode ne peut être érigée en système, car la plus légère modification des questionnaires suffit à bouleverser les règles empiriquement établies. (2) La méthode biométrique de R. A. Fisher. Elle consiste à faire certaines mesures sur des prélèvements effectués à intervalles réguliers et de les comparer avec des corrélations établies antérieurement. C'est le *precision record scheme* utilisé depuis 1928 et extensivement depuis 1933.

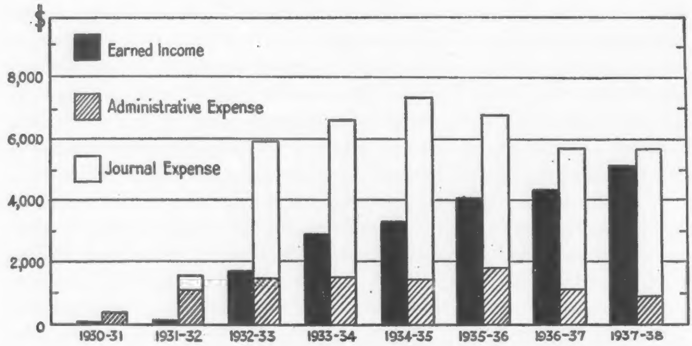
Cependant l'un et l'autre procédés sont sujets aux erreurs résultant de l'*influence retardée* du temps sur les récoltes. On en a tenu compte en recherchant les "périodes critiques" de la croissance. Mais l'effort le plus soutenu a été développé dans la détermination des facteurs météorologiques qui ont une action sur la récolte. Les premiers essais sont dus à Hooker (1907, 1922) qui a recherché l'action de la température et des pluies sur les récoltes. A citer également les travaux de Walter (1910, pluies et température sur la canne à sucre à l'île Maurice), Axel Walen (Suède, céréales), et J. Warren Smith. Mais l'essai fondamental a été celui de H. L. Moore.[6] Depuis cette étude, les travaux réputés de J. B. Kincer et de Ezekiel ont vu le jour; il convient également d'évoquer les recherches de Bradford B. Smith (coton), Henney (blés d'hiver, Kansas), Hodges (maïs). De tous les essais, celui de R. A. Fisher est le plus remarquable, car il recherche l'action des facteurs météorologiques dans leur séquence dans l'année et non plus sous la forme de moyennes annuelles. Il n'a pas été possible à M. Samson d'en estimer la valeur de prévision, mais il l'a recherché pour les travaux de Hodges (portant sur la période 1891–1928) sur l'intervalle 1891–1935: les résultats lui ont paru excellents; moins satisfaisants pour les prévisions de Kincer. Résumant l'apport de la méthode de corrélation, il juge ses résultats d'autant meilleurs que les prévisions sont précoces, qu'il y a un facteur météorologique dominant, que le climat est tropical (simplicité du type météorologique), qu'il s'agit de céréales (vins en France, 10 variables nécessaires). En un mot cette méthode est complémentaire de la méthode de l'estimation. M. WIŚNIEWSKI est intervenu pour souligner certains passages de ce remarquable rapport.

Caisse nationale de crédit
aux départements et aux communes
Paris

[6] *Forecasting the Yield and the Price of Cotton*, New York, 1917.

THE ECONOMETRIC SOCIETY

SUMMARY OF ACCOUNTS, 1930–1938



Years ending Sept. 30	Expenses			Earned Income	Deficit
	Administration	Journal	Total		
1930–31	$ 405.44		$ 405.44	$ 27.00	$ 378.44
1931–32	1,069.86	$ 493.00	1,562.86	165.94	1,396.92
1932–33	1,486.50	4,456.26*	5,942.76*	1,774.19	4,168.57*
1933–34	1,508.82	5,054.98	6,563.80	2,866.68	3,697.12
1934–35	1,422.43	5,927.16	7,349.59	3,280.46	4,069.13
1935–36	1,809.32	4,972.27	6,781.59	4,083.73	2,697.86
1936–37	1,044.01	4,684.48	5,728.49	4,371.71	1,356.78
1937–38	919.60	4,764.34	5,683.94	5.151.67	532.27

* Expenses for only the January, April, and July numbers of ECONOMETRICA are included in 1932–33 due to the fact that this was the first year of the Journal and the fiscal year ends September 30. If the October number had been added, the total expenses for the year would have been increased by approximately $700, to a figure about $100 in excess of the $6,563.80 reported for 1933–34.

The total deficit of $18,297.09 has been met in part by donations totalling $15,368.50. The remaining deficit of $2,928.59, together with a working balance of cash and other current net assets amounting to $71.41, is represented by notes of the Society outstanding to the amount of $3000. As security against this liability the Society, in addition to the cash balance, has a stock of approximately 400 complete files of ECONOMETRICA, Vols. 1–6, and about 3400 extra copies of various issues.

The deficit for 1937–38 was a little over $500. A net addition of about 100 members and subscribers in 1938–39 would balance the budget.

ALFRED COWLES 3RD
Treasurer

Colorado Springs

OFFICERS AND COUNCIL FOR 1939

The Council of the Econometric Society has elected the following officers for 1939: Arthur L. Bowley, President; Joseph A. Schumpeter, Vice-President; and Alfred Cowles 3rd, Secretary and Treasurer.

The Council members elected by the Fellows for terms to expire in December, 1941, are Albert Aupetit, Arthur L. Bowley, and Wladyslaw Zawadzki. Council members whose terms expire in December, 1939, are Costantino Bresciani-Turroni, Irving Fisher, Charles F. Roos, and Joseph A. Schumpeter. Those whose terms expire in December, 1940, are Ragnar Frisch, John Maynard Keynes, and F. Zeuthen. Alfred Cowles 3rd, because of his position as Secretary and Treasurer, is an ex officio member of the Council.

ALFRED COWLES 3RD
Secretary

Colorado Springs

OBITUARY

ECONOMETRICA records with deep regret that the deaths of the following members of the Econometric Society have occurred since the last notice was published, in the April, 1937 issue.

Professor Edgard Allix
Faculté de Droit
L'Université de Paris
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Paris XVI, France

Mr. Ogden L. Mills
15 Broad Street
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Dr. Warren M. Persons
27 Everett Street
Cambridge, Massachusetts

M. Gaston Roulleau
Controleur Général de la Banque de France
2 rue Radziwill
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Dr. Karl Schlesinger
Alserbachstrasse 16
Wien IX, Germany

Professor Henry Schultz
University of Chicago
Chicago, Illinois

Professor G. F. Warren
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